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Numerigal Predications of Turning Points of Nonlinear Problems*

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Abstract A numerieal predication of turning points of a nonlinear problem and its finite element approximate problem is discussed and applied to the stationary Navier-Stokes equations.

§1. Introduction

Let X be a Banach space and $F : \mathbf{R} \times X \to X$ be a bounded C^m-mapping. We consider a parameter dependent equation

$$F(\lambda, u) = 0. \tag{1.1}$$

Such problems are often encountered in physics and chemistry. Usually, the parameter λ represents some physical quantities, e.g. temperature, density, Reynolds number and force, etc. (cf. Brezzi/Rappaz/Raviart [1,2]).

Bifurcation theory and its numerical analysis are concerned with the existence and the structure of multiple solution curves of (1.1) and their numerical approximations. As a text for the "State-of-the-Art", one can see the proceedings Mittelmann/Weber [15], Küpper/Mittelmann/Seydel [7], Küpper/Seydel/Troger [8], Li et al [11] and Mittelmann/Roose [16].

Concerning the turning point problems of (1.1), if the existence of a turning point on the solution curve is already known, then special methods can be employed to approximate it, see for example Melhem/Rheinboldt [13]. Nevertheless, usually it is prior unknown in the numerical path following whether turning points exist. In fact, conjectures of the existence of the turning points come mostly from numerical results. In this paper we will investigate the behaviour of the solution of (1.1) on the basis of its discretizations and show a method for predicting the existence of turning points of a quadratic problems in its numerical approximations. The results presented here is a generalization of Scholz [17]. An outline is as follows.

Section §2 shows briefly the existence of nonsingular solution of (1.1). The existence and distribution of the turning points of (1.1) are discussed in Section §3. The stationary

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Navier-Stokes equations are considered in Section §4 with the penalty finite element methods.

§2. Existence of the Nonsingular solutions

Let $X_h \subset X$ and $F_h : \mathbf{R}\times X_h \to X_h$ be approximations of the space X and the mapping F respectively. Furthermore, we assume that F_h is C^m-continuous and its definition domain can be continuously exteneded to the whole space $\mathbf{R}\times X$ (cf. Brezzi/Rappaz/Raviart [1]). Consider the equation

$$F_h(\lambda, u) = 0 \tag{2.1}$$

in the space $\mathbf{R}\times X_h$. For the reason of convenience, solutions of (2.1) on $\mathbf{R}\times X$ are supposed to be in $\mathbf{R}\times X_h$. Otherwise, the operator F_h in (2.1) can be replaced by $F_h(\cdot, \Pi_h \cdot)$ with Π_h as a projection from X onto X_h. Furthermore, we assume

H1. For $j = 0, 1$

$$\lim_{h\to 0} \sup_{(\lambda,u)\in\Lambda\times E} \|D_u^j(F - F_h)(\lambda, u)\|_{L_j(X,X)} = 0; \tag{2.2}$$

H2. For $j \in \{0, 1, \cdots, m\}$,

$$\sup_{h>0} \sup_{(\lambda,u)\in\Lambda\times E} \|D^j(F - F_h)(\lambda, u)\|_{L_j(\mathbf{R}\times X,X)} < +\infty, \tag{2.3}$$

where $\Lambda\times E$ is an arbitrary bounded subset of $\mathbf{R}\times X$ and $L_j(X, X)$ is the space of bounded j-multiple linear mappings from X into X, and $L(X, X) = L_1(X, X)$.

Theorem 2.1 *Let the conditions* **(H1)**, **(H2)** *be satisfied and*

(i) *For all $h > 0$, equation (2.1) has a solution (λ_h, u_h) and the solution sequence $\{(\lambda_h, u_h), h > 0\}$ is uniformly bounded;*

(ii) *There is a constant $d > 0$ independent of λ, such that*

$$\|D_u F(\lambda_h, u_h)v\|_X \ge d\cdot\|v\|_X, \ \forall v \in X, h > 0. \tag{2.4}$$

Then, there are constants $\alpha, \beta, h_0, k_0 > 0$, such that for all $h \in (0, h_0]$, the equation (1.1) has a unique solution branch $\{(\lambda, u(\lambda)); |\lambda - \lambda_h| \le \alpha\}$ in the neighborhood $S((\lambda_h, u_h); \alpha, \beta)$ of (λ_h, u_h) and

$$\|u(\lambda) - u_h\|_X \le K_0\cdot\{|\lambda - \lambda_h| + \|F(\lambda_h, u_h)\|_X\}, \tag{2.5}$$

where $((\lambda_h, u_h); \alpha, \beta) := \{(\lambda, u) \in \mathbf{R}\times X, |\lambda - \lambda_h| \le \alpha, \|u - u_h\|_X \le \beta\}$.

Proof From the condition (ii), we know that the operator $D_u F^{-1}(\lambda_h, u_h)$ is well defined and is uniformly bounded with $d^{-1} > 0$. Furthermore, condition (i) shows

$$\sup_{h>0} \|D^m F(\lambda_h, u_h)\|_{L_m(\mathbf{R}\times X,X)} < +\infty \tag{2.6}$$

and

$$\|D_uF(\lambda,u)-D_uF(\lambda_h,u_h)\|\le C\cdot\|(\lambda,u)-(\lambda_h,u_h)\|. \tag{2.7}$$

Moreover,

$$\lim_{h\to 0}\sup_h\|F(\lambda_h,u_h)\|=\lim_{h\to 0}\sup_h\|(F-F_h)(\lambda_h,u_h)\|_X=0. \tag{2.8}$$

Since the statements (2.6)-(2.8) hold for all $h>0$, our conclusions follow directly from Theorem 1 in Brezzi/Rappaz/Raviart [2].

Rmark 2.1 Every accomulating point (λ_0,u_0) of $\{(\lambda_h,u_h);h>0\}$ is a solution of (1.1) and it is regular if and only if for any subsequence $\{(\lambda_{h'},u_{h'});h'>0\}$ converging to (λ_0,u_0), we have

$$\lim_{h'\to 0}\|D_uF(\lambda_{h'},u_{h'})\|_{L(X,X)}>0. \tag{2.9}$$

§3. Numerical Predication of Nondegenerate Turning Points

In this section we consider a determination of turning points of a quadratic problem with its descrete solutions. To this end, let us assume that X is a functional Hilbert space and Y is a Banach space. We denote $(\cdot,\cdot)$ the inner product in X with the corresponding norm

$$\|u\|_X:=(u,u)^{1/2}.$$

Suppose $T\in L(Y,X)$ is a compact mapping and $G:X\to Y$ is a homogeneous quadratic mapping. Furthermore, there are approximations $X_h\subset X$ for X and $T_h\in L(Y,X_h)$ for T and

$$\lim_{h\to 0}\|T-T_h\|_{L(Y,X)}=0. \tag{3.1}$$

In the sequel, $C,C_0,\cdots,K_0,K_1,\cdots$ will represent various constants independent of λ,u,h etc. We consider the equation

$$F(\lambda,u):=\lambda u-T[G(u)+g]=0 \tag{3.2}$$

and its approximations

$$F_h(\lambda_h,u_h):=\lambda_h u_h-T_h[G(u_h)+g]=0. \tag{3.3}$$

Remark 3.1 From the definition of T_h,F_h, one sees that the equation (3.3) can be considered on the whole space $\mathbf{R}\times X$ and its solution remains always in $\mathbf{R}\times X_h$.

We make the following hypotheses.

A1) For $h>0$, equation (3.3) has a solution (λ_h,u_h) and the solution sequence $\{(\lambda_h,u_h);h>0\}$ is uniformly bounded;

A2) λ_h is an eigenvalue of $T_h DG(u_h)$ on $X_h (h > 0)$, i.e., there is at least one element $\eta_h \in X_h, \|\eta_h\|_X = 1$, such that

$$\lambda_h \eta_h - T_h D_u G(u_h)\eta_h = 0. \tag{3.4}$$

Obviously, there is also $\eta_h^* \in X_h, \|\eta_h^*\|_X = 1$ satisfying

$$\lambda_h \eta_h^* - [T_h D_u G(u_h)]^* \eta_h^* = 0; \tag{3.5}$$

A3) For the decomposition $X = X_1 + X_2$ with $X_2 := \text{span}[\eta_h], X_1 = X_2^{\perp}$, there is a constant $d > 0$ independent of $h > 0$, such that

$$\|\lambda_h v - TD_u G(u_h)v\|_X \geq d \cdot \|v\|_X, \ \forall v \in X_1, h > 0; \tag{3.6}$$

A4) There are constants $\varepsilon \in [0, 1/5], q > 0$ independent of $h > 0$, such that

$$d \cdot \min\{|\alpha_h|, |\beta_h|\} \geq q \cdot \|T - T_h\|_{L(Y,X)}^{\varepsilon}, h > 0, \tag{3.7}$$

where $\alpha_h := (\eta_h^*, TG(u_h)), \beta_h := (\eta_h^*, u_h)$;

A5) For each $u \in X$, we have

$$\|G(u)\|_Y \leq K_0 \cdot \|u\|_X^2, \ \|D_u G(u)\|_{L(X,Y)} \leq K_0 \cdot \|u\|_X. \tag{3.8}$$

Lemma 3.1 *$\eta_h(\eta_h^*)$ is an approximate eigenfunction of $TD_u G(u_h)([TD_u G(u_h)]^*)$ w.r.t λ_h. More precisely,*

$$\|\lambda_h \eta_h - TD_u G(u_h)\eta_h\|_X \leq C_1 \cdot \|T - T_h\|_{L(Y,X)}.$$

Under **A3)**, every element w in X has a unique decomposition:

$$w = z + \xi\eta_h, \ z \in X_1, \ \xi \in \mathbf{R}. \tag{3.9}$$

We define a mapping $H : X \to X$ by

$$Hw := TD_u G(u_h)z + \lambda_h \xi \eta_h. \tag{3.10}$$

Since T is compact, so $H \in L(X, X)$. Furthermore,

$$\lambda_h \eta_h - H\eta_h = 0. \tag{3.11}$$

On the other hand,

$$Hw = TD_u G(u_h)w, \ \forall w \in X_1. \tag{3.12}$$

Hence, it follows from (3.6) that

$$\|\lambda_h w - Hw\|_X \geq d \cdot \|w\|_X, \ \forall w \in X_1. \tag{3.13}$$

That means, λ_h is a simple eignevalue of H (and H^*) with the eigenfunction η_h (and $\tilde{\eta}_h^*$) normalized by $\|\eta_h\|_X = 1$. Directly calculating shows

$$(H - TD_uG(u_h))w = \xi(\lambda_h\eta_h - TD_uG(u_h)\eta_h),\ \forall w \in X.$$

Then, Lemma 3.1 implies

$$\|H - TD_uG(u_h)\|_{L(X,X)} \le C_1\cdot\|T - T_h\|_{L(Y,X)}. \tag{3.14}$$

Lemma 3.2 *For the functions $\eta_h^*, \tilde{\eta}_h^*$ above, we have*

$$\|\eta_h^* - \tilde{\eta}_h^*\|_X \le C_2\cdot d^{-1}\cdot\|T - T_h\|_{L(Y,X)}. \tag{3.15}$$

Proof We decompose the space X into $X = X^1 + X^2$ with $X^2 := \text{span}[\tilde{\eta}_h^*]$ and $X^1 = (X^2)^\perp$. It holds that (see Lemma 3.1 and (3.13)):

$$\|\lambda_h\eta_h^* - [TD_uG(u_h)]^*\eta_h^*\|_X \le C_1\cdot\|T - T_h\|_{L(Y,X)}$$

and

$$\|\lambda_h w - Hw\|_X \ge d\cdot\|w\|_X,\ \forall w \in X^1.$$

In particular, for $w := \eta_h^* - \xi\tilde{\eta}_h^*, \xi := (\eta_h^*, \tilde{\eta}_h^*)$, we have $w \in X^1$ and

$$d\cdot\|w\|_X \le 2C_1\cdot\|T - T_h\|_{L(Y,X)}.$$

Therefore,

$$|1 - \xi| \le 4d^{-1}\cdot q^{-1}\cdot C_1\cdot\|T - T_h\|_{L(Y,X)}^{2-\epsilon}.$$

Finally

$$\begin{aligned} \|\eta_h^* - \tilde{\eta}_h^*\|_X &\le \|\eta_h^* - \xi\tilde{\eta}_h^*\| + \|\xi\tilde{\eta}_h^* - \tilde{\eta}_h^*\|_X \\ &\le 2C_1(1 + 2q^{-1}C_1)d^{-1}\cdot\|T - T_h\|_{L(Y,X)}. \end{aligned}$$

By choosing $C_2 := 2C_1(1 + 2q^{-1}C_1)$, we complete the proof.

Denote

$$\alpha_h' := (\eta_h^*, TG(\eta_h)), \beta_h' := (\eta_h^*, u_h). \tag{3.16}$$

We have

Lemma 3.3 *α_h', β_h' are approximations of α_h, β_h respectively and*

$$\max\{|\alpha_h - \alpha_h'|, |\beta_h - \beta_h'|) \le C_1 d^{-1}\cdot\|T - T_h\|_{L(Y,X)}.$$

The mapping $\lambda_h I - H$ satisfies

$$\text{Range}(\lambda_h I - H) = X_1$$

and is invertible on the space X_1. We denote its inverse by $\tilde{A}$. Let P be the orthogonal projection from X onto X_1 and $\tilde{A} = A\cdot P$. Then

$$A \in L(X,X) \text{ and } \|A\|_{L(X,X)} \le d^{-1}. \tag{3.17}$$

Lemma 3.4 *$w \in X$ is a solution of the equation*

$$(\lambda_h I - H)w = z \tag{3.18}$$

if and only if there is an element $z \in X, \xi \in \mathbf{R}$, such that

$$w := Az + \xi\eta_h, \ (\tilde{\eta}_h^*, z) = 0. \tag{3.19}$$

Proof Let w be a solution of (3.18). Then $A(\lambda_h I - H)w = Az$. According to the definition of mapping A, there is $\xi \in \mathbf{R}$, such that

$$w := Az + \xi\eta_h, \ (\tilde{\eta}_h^*, z) = (\tilde{\eta}_h^*, (\lambda_h I - H)w) = 0.$$

On the other hand, if (3.19) holds, Fredholm operator theory shows that there exists at least one $w_0 \in X$ such that $(\lambda_h I - H)w_0 = z$. The fact $\mathrm{Null}(\lambda_h I - H) = \mathrm{span}\{\eta_h\}$ and the compactness of H show that w and w_0 are consistent in the sense of difference of η_h. □

Let us go back to the problem (3.1) and replace the variables in it by

$$\lambda = \lambda_h + \sigma, \quad u = u_h + v, \tag{3.20}$$

where $v \in X$ and $\sigma \in \mathbf{R}$ with $|\sigma| \leq \Delta\lambda_h := |\alpha_h \beta_h^{-1}| \cdot \|T - T_h\|^{3\epsilon}$. Then the problem (3.2) becomes

$$(\lambda_h I - H)v = TG(v) + (TD_uG(u_h) - H)v - \sigma(u_h + v) + (T - T_h)(g + G(u_h)). \tag{3.21}$$

This equation is equivalent to that there is a $\xi \in \mathbf{R}$, such that

$$\begin{aligned} v &= A[TG(v) + (TD_uG(u_h) - H)v - \sigma(u_h + v) + (T - T_h)(G(u_h) + g)] + \xi\eta_h \\ &= : B(\sigma, \xi, v) \end{aligned} \tag{3.22}$$

and

$$(\tilde{\eta}_h^*, TG(v) + (TD_uG(u_h) - H)v - \sigma(u_h + v) + (T - T_h)(G(u_h) + g)) = 0. \tag{3.23}$$

Denote $\rho := \|T - T_h\|_{L(Y,X)}^{3\epsilon/2}, \rho_0 := \rho_h/2$. We have

Lemma 3.5 *For $\xi \in S(0, \rho), \sigma \in S(0, \Delta\lambda_h)$, there is a constant $h_0 > 0$, such that for all $h \in (0, h_0]$, the mapping $B(\sigma, \xi, \cdot)$ is contractive from $S_X(0, \rho)$ into itself.*

For each $(0, \xi) \in S(0, \Delta\lambda_h) \times S(0, \rho_0)$, from Lemma 3.5 one gets a unique fixed point $v = v(\sigma, \xi)$ of $B(\sigma, \xi, \cdot)$ from (3.22). To parametrize it with ξ, we consider the problem (3.23) for a given $\sigma \in S(0, \Delta\lambda_h)$. First of all, we decompose the fixed point $v = B(\sigma, \xi, v)$ into

$$v = \xi\eta_h + v_2, v_2 \in X_1 \tag{3.24}$$

and

$$\|v_2\|_X \leq Cd^{-1} \cdot \|T - T_h\|_{L(Y,X)}^{3\epsilon}.$$

Hence, the equation (3.23) becomes

$$q(\sigma,\xi) := \xi^2 + a\sigma + b(\sigma,\xi) = 0, \tag{3.25}$$

where $a := -\beta'_h/\alpha'_h$ and

$$b(\sigma,\xi) := {\alpha'_h}^{-1}\cdot(\tilde{\eta}^*_h, \xi T D_u G(u_h)v_2 + TG(v_2) + (TD_uG(u_h) - H - \sigma I)v + (T - T_h)(g + G(u_h))).$$

without loss of the generality, we assume $\text{sign}(\alpha_h\cdot\beta_h) = -1$. Then $a > 0$.

Examining the properites of the mapping $B(\sigma,\xi,\cdot)$, we get

Lemma 3.6 *There is a constant $h_0 > 0$, such that for all $h \in (0, h_0]$, the function $q(\sigma,\xi)$ in (3.25) has the following properties.*

a) *Fro a given $\sigma \in S(0, \Delta\lambda_h)$, there exists uniquely $\xi(\sigma) \in S(0,\rho_0)$. The function $q(\sigma,\cdot)$ takes a minimum at $\xi(\sigma)$ and is monotone decrease in $[-\rho_0, \xi(\sigma)]$ and monotone increase in $(\xi(\sigma), \rho_0]$;*

b) *For $p(\sigma) := \min\limits_{\xi} q(\sigma,\xi)$, there is $\sigma^* \in S(0,\Delta\lambda_h)$, such that $p(\sigma^*) = 0$;*

c) *For $\sigma = \sigma^*, q(\sigma^*,\xi)$ has a unique root. For each $\sigma \in [-\Delta\lambda_h, \sigma^*), q(\sigma,\xi)$ has exactly two roots $\xi^-(\sigma) < \xi^+(\sigma)$ in $S(0,\rho_0)$. For each σ in $(\sigma^*, \Delta\lambda_h]$, we have $q(\sigma,\xi) > 0, \forall\xi \in S(0,\rho_0)$.*

d) *$\xi^-(\sigma), \xi^+(\sigma)$ are all continuous functions and*

$$\lim_{h\to\sigma^*} \xi^-(\sigma) = \lim_{h\to\sigma^*} \xi^+(\sigma) = \xi^*.$$

Theorem 3.1 *If the conditions **A1)-A5)** are satisfied and $\text{sign}(\alpha_h\beta_h) = -1$, there are contants $h_0 > 0, \sigma^* \in S(0,\Delta\lambda_h)$, such that for $h \le h_0$, we have*

1) For $\lambda = \lambda_h + \sigma,\ \sigma \in [-\Delta\lambda_h, \sigma^*)$, equation (3.1) has two real solutions $u^-(\lambda), u^+(\lambda)$ of (3.1) in $S_X(u_h,\rho)$;

2) For $\lambda^* = \lambda_h + \sigma^*$, equation (3.1) has a unique solution u^* in $S_X(u_h,\rho)$ and

$$\lim_{\lambda\to\lambda^*} u^-(\lambda) = \lim_{\lambda\to\lambda^*} u^+(\lambda) = u^*;$$

3) For $\lambda = \lambda_h + \sigma, \sigma \in (\sigma^*, \Delta\lambda_h]$, equation (3.1) has no solutions in $S_X(u_h,\rho)$.

Proof Since the problem (3.2) is equivalent to (3.22), (3.23), one solves a function $v(\sigma,\xi)$ as a fixed point of $B(\sigma,\xi,\cdot)$ from (3.22) and substitute it into (3.23). In this way, the problem (3.2) is reduced to an algebraic equation (3.25) for σ,ξ. At the same time, Lemma 3.6 shows that $q(\sigma,\xi)$ is mainly a parabolic type of function and its roots $(\sigma,\xi(\sigma))$ has exactly the above structure 1)-3) which determine also the srtucture of the solution $(\lambda, u(\lambda))$ of (3.2), see Fig. 1.

Remark 3.2 If $\mathrm{sign}(\alpha_h\beta_h)=1$ and the conditions A1)-A5) are satisfied, then the above conclusion 2) still holds, while 1) and 3) hold for σ in $(\sigma^*,\Delta\lambda_h]$ and $[-\Delta\lambda_h,\sigma^*)$ respectively.

The conditions **A1),A5)** are rather trivial. Conversely, to verify the conditions **A2)-A4)**, one can observe the exchange of signs of determinant of the Jacobian matrix $\lambda_h I - T_h DG(u_h)$ or investigate its rank deficiency during the numerical computations, see e.g. Chan [3].

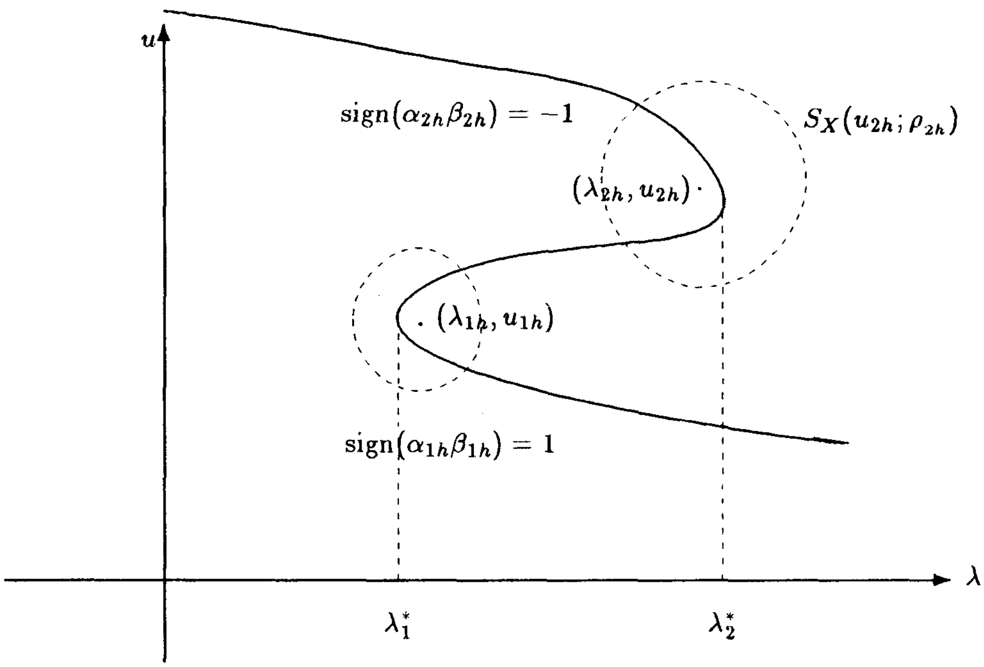

Fig.1

§4. Application to Stationary Navier-Stokes Equations

In this section we will apply the previous results to the stationary Navier- Stokes equations:

$$\begin{cases} -\lambda \nabla_j \nabla^j u^i + u^j \nabla_j u^i + \nabla^i p &= f^i, \text{ in } \Omega\subset \mathbf{R}^n, i=1,\cdots,n \\ \mathrm{div}(u) &= 0, \text{ in } \Omega, \\ u &= 0, \text{ on } \partial\Omega, \end{cases} \tag{4.1}$$

where $n=2$ or $3, \lambda = Re^{-1}$ is the Reynolds number, u,p are velocity and pressure of the fluid respectively. We will consider the product spaces $X:=[H^1(\Omega)]^n$ and $X_0:=[H_0^1(\Omega)]^n, M:=L_0^2(\Omega)=\{q\in L^2(\Omega)|\int_\Omega q dx=0\}$. Furthermore, we introduce two bilinear and one trilinear forms:

$$a_0(u,v):=\int_\Omega \nabla u\cdot\nabla v dx = \int_\Omega \nabla_i u^j\cdot\nabla^i v_j dx,\quad \forall u,v\in X,$$

$$a_1(u;w,v) : = \int_\Omega u^j\cdot \nabla_j w^i\cdot v_i dx, \qquad \forall u,v,w \in X,$$
$$b(u,q) : = -\int_\Omega q\cdot \mathrm{div}(u) dx, \qquad \forall q \in M, u \in X.$$

Obviously, $a_0(\cdot,\cdot)$ is continuous, symmetric and coercive on $X_0 \times X_0$. Moreover, $a_1(\cdot;\cdot,\cdot)$ is continuous on $X \times X \times X$ and $b(\cdot,\cdot)$ is also continuous on $X \times M$ (cf. Li [9,10]). A primitive variable variational formulation of the stationary Navier-Stokes problem (4.1) is

$$\begin{cases} \text{Find } (u,p) \in X_0 \times M, \text{ such that} & \\ \lambda a_0(u,v) + a_1(u;u,v) + b(v,p) = \langle f, v\rangle, & \forall v \in X_0, \\ b(u,q) = 0, & \forall q \in M, \end{cases} \tag{4.2}$$

where $\langle\cdot,\cdot\rangle$ is the dual product on $X_0 \times X_0'$ and X_0' is the dual space of X_0. It is well known that equation (4.2) has at least one solution (cf. Temam [18, 19]). We denote $S(f,\lambda) \subset X_0 \times \mathbf{R}^+$ the solution set of (4.2) and $S(f) := \cup_{\lambda>0} S(f,\lambda)$. Then there exists a G_δ-dense set $D \subset L^2(\Omega)$, such that for every $f \in D$, the solution set $S(f)$ is one dimensional C^∞-manifold (cf. Temam [19]).

In order to eliminate the constraint condition $\mathrm{div}(u) = 0$, i.e., the incompressibility, we make use of a penalty method (cf. Li [9, 10]). Assume

$$p = -1/\varepsilon\cdot \mathrm{div}(u),\ \varepsilon \in \mathbf{R}^+ \tag{4.3}$$

and take it into (4.2). The problem (4.2) becomes

$$\begin{cases} \text{Find } u \in X_0, \text{ such that} \\ \lambda A(u,v) + a_1(u;u,v) = \langle f,v\rangle,\ \forall v \in X_0, \end{cases} \tag{4.4}$$

where $A(u,v) := a_0(u,v) + \sigma\cdot(\mathrm{div}(u),\mathrm{div}(v)), (\cdot,\cdot)$ is the inner product on $L^2(\Omega)$ and $\sigma = 1/\varepsilon\lambda \in \mathbf{R}^+$ is a arbitrary fixed small constant owing to ε being arbitrary positive constant. It is obviously that $A(\cdot,\cdot)$ is a symmetric, coercive and bounded bilinear form on $X_0 \times X_0$ and $A(u,u) \geq |u|^2_{1,\Omega}$, $\forall v \in X_0$. On the other hand, (4.4) can be rewritten into an operator form. In fact, for all $u \in X_0$, there exists a $G(u) \in Y := [L^{4/3}(\Omega)] \subset X_0'$, such that

$$\langle G(u), v\rangle = a_1(u;u,v), \quad \forall v \in X_0. \tag{4.5}$$

Moreover, the mapping $G(u)$ has the following properties:

$$||G(u)||_{0,4/3} \leq N\cdot|u|_1^2, \qquad \forall u, \in X_0, \tag{4.6}$$
$$||G(u) - G(v)||_{0,4/3} \leq N\cdot(|u|_1 + |v|_1)|u - v|_1, \forall u,v \in X_0, \tag{4.7}$$
$$\langle D_u G(u)w, v\rangle = a_1(u;w,v) + a_1(w;u,v), \qquad \forall u,v,w \in X_0, \tag{4.8}$$
$$||D_u G(u)||_{L(X_0,Y)} \leq 2N\cdot|u|_1, \qquad \forall u \in X_0, \tag{4.9}$$
$$G(u+w) = G(u) + G(w) + D_u G(u)w = G(u) + G(w) + D_u G(w)u, \tag{4.10}$$

where $D_u G$ is the Frechet derivative of G and

$$N := ||a_1|| = \sup_{|u|_1 = |v|_1 = |w|_1 = 1} |a_1(u;w,v)|.$$

By virtue of regularity properties of solutions of Stokes problems (cf. Temam [18]), one can define an operator $T : X_0' \to X_0$ by

$$A(Tg, v) = \langle g, v \rangle, \quad \forall v \in X_0. \tag{4.11}$$

Moreover, the operator T maps $[L^r(\Omega)]^n$ into $[H^{2,r}(\Omega) \cap H_0^1(\Omega)]^n$ (see Temam [18]). For $f \in Y$, we obtain an operator form of (4.4):

$$\lambda u + TG(u) = \tilde{f}, \tag{4.12}$$

where $\tilde{f} := Tf$. Since for all $u \in X_0$,

$$TG(u) \in [H^{2,4/3}(\Omega) \cap H_0^1(\Omega)]^n \subset X_0,$$

the operator $TG(\cdot)$ is compact from X_0 into X_0. Similarly, its derivative $TD_uG(u)$ is also compact from X_0 into X_0. Consequently, $\lambda I - TD_uG$ is a Fredholm operator with the index 0.

Let $X_h \subset X_0$ be a finite approximation of X_0 and $h > 0$ be a parameter of discretizations. For h small enough, we assume that there is an integer $m \in \mathbf{N}$, such that

$$|u - \Pi_h u|_1 \leq C \cdot h^m, \ \forall u \in X_0, \tag{4.13}$$

where Π_h is the orthogonal projection from X_0 onto X_h under the seminorm on X_0. An approximate operator $T_h : Y \to X_0$ of T is defined by:

$$A(T_h g, v) = \langle g, v \rangle, \quad \forall v \in X_h. \tag{4.14}$$

We denote $u_h := T_h g$. Then, for the solution u of (4.11) and u_h of (4.14) we have (cf. Girault/Raviart [4]):

$$|u - u_h|_1 \leq \inf_{v \in X_h} |u - v|_1 \leq |u - \Pi_h u|_1 \leq C \cdot h^m.$$

Hence,

$$\|T - T_h\| \leq C \cdot h^m. \tag{4.15}$$

As usual, an approximate problem of (4.4) is defined by

$$\begin{cases} \text{Find } u_h \in X_h, \text{ such that} \\ \lambda A(u_h, v) + a_1(u_h; u_h, v) = \langle f, v \rangle, \ \forall v \in X_h. \end{cases} \tag{4.16}$$

We consider here an operator form (see (4.12)):

$$\lambda u_h + TG(u_h) = \tilde{f}. \tag{4.17}$$

If u, u_ε and u_h are solutions of (4.1), (4.4) and (4.16) respectively, the statements (4.6)-(4.10) and Theorem 3.3 in Chapter IV, Girault/Raviart [4] yield:

$$\begin{aligned} |u_\varepsilon - u_h|_1 &\leq K \cdot \|T - T_h\| \cdot (\|G(u_\varepsilon)\| * + \|f\| *)/\lambda \\ &\leq K/\lambda \cdot (N|u_\varepsilon|_1^2 + \|f\| *) \cdot \|T - T_h\| |u - u_\varepsilon|_1 \\ &\leq C \cdot \varepsilon, \end{aligned}$$

where $K, C > 0$ are constants independent of $h > 0, u_h, \lambda$, and $\|\cdot\|*$ denotes the dual norm of $|\cdot|_1$. Therefore, we have

$$|u - u_h|_1 \le C\cdot\varepsilon + K\cdot(N|u_\epsilon|_1^2 + \|f\|*)\cdot Re\cdot\|T - T_h\|. \tag{4.18}$$

Assume that (4.17) has a solution u_h and λ_h is a simple eigenvalue of $-T_h D_u G(u_h)$, i.e., there is $\varphi_h \in X_h$ with $|\varphi_h|_1$, such that

$$\lambda_h A(\varphi_h, v) + A(u_h; \varphi_h, v) = 0, \forall v \in X_h, \tag{4.19}$$

where $a(u; w, v) := a_1(u; w, v) + a_1(w; u, v)$. Its operator form is

$$[\lambda_h I + T_h D_u G(u_h)]\varphi_h = 0. \tag{4.19$'$}$$

The eigenvector ψ_h of the adjoint operator $[T_h D_u G(u_h)]^*$ is defined by

$$\lambda_h A(\psi_h, v) + a(v; u_h, \psi_h) = 0, \forall v \in X_h \tag{4.20}$$

and

$$A(\varphi_h, \psi_h) = 1. \tag{4.21}$$

Thereafter, we do the decomposition $X = X_1 \dot{+} X_2$ with

$$X_2 := \mathrm{span}[\varphi_h], X_1 := X_2^{\perp} = \{u \in X_0 | A(u, \psi_h) = 0\}. \tag{4.22}$$

Define

$$\begin{cases} \alpha_h := A(TG(\varphi_h), \psi_h) = \langle G(\varphi_h), \psi_h\rangle = a_1(\varphi_h; \varphi_h, \psi_h) \\ \beta_h := A(u_h, \varphi_h). \end{cases} \tag{4.23}$$

From the hypothesis A4), we have $\alpha_h \ne 0$ and $\beta_h \ne 0$. Let us denote

$$\lambda = \lambda_h + \sigma,\ u = u_h + \xi\varphi_h + z,\ z \in X_1 \tag{4.24}$$

and substitute them into (4.17). It follows

$$(\lambda_h + \sigma)(u_h + \xi\varphi_h + z + TG(u_h + \xi\varphi_h + z) = \tilde{f}.$$

Utilizing the statements (4.10), (4.17) and (4.19), one derives

$$\begin{aligned}(\lambda_h + \sigma)z + [TD_uG(u_h) + \xi TD_uG(\varphi_h)]z + TG(z) = \tilde{f} - \tilde{f}_h + (T_h - T)G(u_h) \\ + \xi(T_h - T)D_uG(u_h)\varphi_h - \sigma u_h - \xi\sigma\varphi_h - \xi TG(\varphi_h).\end{aligned} \tag{4.25}$$

Lemma 3.5 shows that the equation (4.25) has a unique solution $z = z(\sigma, \xi)$. Noticing the statements (4.23) and

$$A(z, \psi) = 0, A(TG(\cdot), \cdot) = a(\cdot; \cdot, \cdot), A(TD\quad G(\cdot)\cdot, \cdot) = a(\cdot, \cdot, \cdot),$$

we obtain a discrete bifurcation equation (see (3.23), (3.28)):

$$q(\sigma,\xi) := a(u_h; z, \psi_h) + \xi a(\varphi_h; z, \psi_h) + a(z; z, \psi_h)$$

$$\begin{aligned}&-A((T-T_h)f,\psi_h) - A((T_h - T)G(u_h),\psi_h)\\ &-\xi A((T_h-T)D_uG(u_h)\varphi_h,\varphi_h,\psi_h) + \sigma\beta_h + \xi\sigma + \xi^2\alpha_h = 0.\end{aligned} \tag{4.26}$$

If $z(\sigma,\xi)$ has the asymptotic expansion

$$z(\sigma,\xi) = z_0 + z_1\sigma + z_2\xi + z_{11}\sigma^2 + z_{12}\xi\sigma + z_{22}\xi^2 + \cdots, \tag{4.27}$$

we derive from (4.25) various equations for $z_0, z_1, \cdots$, in X_1 as follows:

$$\begin{aligned}
&\text{a)}\ \ \lambda_h z_0 + TD_uG(u_h)z_0 + TG(z_0) = \tilde f - \tilde f_h + (T_h - T)G(u_h),\\
&\text{b)}\ \ Qz_1 = -u_h - z_0,\\
&\text{c)}\ \ Qz_2 = -TD_uG(\varphi_h)z_0 + (T_h - T)D_uG(u_h)\varphi_h,\\
&\text{d)}\ \ Qz_{11} = -TG(z_1) - z_1,\\
&\text{e)}\ \ Qz_{12} = -TD_uG(z_1)z_2 - TD_uG(\varphi_h)z_1 - z_2 - \varphi_h,\\
&\text{f)}\ \ Qz_{22} = -TG(z_2) - TD_uG(\varphi_h)z_2 - TG(\varphi_h),
\end{aligned} \tag{4.28}$$

where $Q := \lambda_h I + TD_uG(u_h) + TD_uG(z_0)$. In view of $A(TG(z_0), z_0) = 0$, we get from (4.28a)

$$d\cdot|z_0|_1^2 \le [\|\tilde f - \tilde f_h\| * + |(T_h - T)G(u_h)|_1]\cdot|z_0|_1,$$

where $d > 0$ is a constant independent of h. The estimate (4.6) provides

$$\begin{aligned}d\cdot|z_0|_1 &\le \|\tilde f - \tilde f_h\| * + \|(T_h - T)\|\cdot N\cdot|u_h|_1^2,\\
&\le \|T - T_h\|(\|f\| * + N\cdot|u_h|_1^2)\\
&\le \|T - T_h\|(\|f\| * + N\cdot\lambda^{-2}\cdot\|f\|^2 *)\\
&\le C_1\cdot h^m.\end{aligned} \tag{4.29}$$

Similarly, from (4.28c) we have

$$|z_2|_1 \le d^{-1}\cdot 2N\cdot(|z_0|_1 + |u_h|_1\cdot\|T - T_h\|) \le C_2 h^m. \tag{4.30}$$

On the other hand, we derive from (4.26), (4.27) that

$$\begin{aligned}
q(0,0) &= a(u_h; z_0, \psi_h) + a_1(z_1; z_0, \psi_h) - A((T - T_h)f, \psi_h) - A((T - T_h)G(u_h), \psi_h),\\
\frac{\partial q}{\partial\sigma}(0,0) &= a(u_h; z_1, \psi_h) + a(z_0; z_1, \psi_h) + \beta_h,\\
\frac{\partial q}{\partial\xi}(0,0) &= a(u_h; z_2, \psi_h) + a(\varphi_h; z_0, \psi_h) + a(z_0; z_2, \psi_h)
\end{aligned}$$

$$- A((T-T_h)D_uG(u_h)\varphi_h, \psi_h),$$

$$\begin{aligned}
\frac{\partial^2 q}{\partial \sigma^2}(0,0) &= a(u_h; z_{11}, \psi_h) + a(z_0; z_{11}, \psi_h) + a_1(z_1; z_1, \psi_h), \\
\frac{\partial^2 q}{\partial \xi^2}(0,0) &= a(u_h; z_{22}, \psi_h) + a(\varphi_h; z_1, \psi_h) + a_1(z_2; z_2, \psi_h) + a(z_0, z_{22}, \psi_h) + \alpha_h, \\
\frac{\partial^2 q}{\partial \xi \partial \sigma}(0,0) &= a(u_h; z_{12}, \psi_h) + a(\varphi_h; z_1, \psi_h) + a(z_1; z_2, \psi_h) + a(z_2, z_{12}, \psi_h) + 1.
\end{aligned}$$

Since for all $w \in X_1, T_hD_uG(u_h)w \in X_1$ holds, we have the equality

$$a(u_h; z, \psi_h) = A((T-T_h)D_uG(u_h)z, \psi_h).$$

Therefore, the function $q(\sigma,\xi)$ in (4.26) can be rewritten as

$$\begin{aligned}
q(\sigma,\xi) &= A((T-T_h)[D_uG(u_h)(z-\varphi_h) - f - G(u_h)], \psi_h) + \xi a(\varphi_h; z, \psi_h) \\
&\quad + a_1(z; z, \psi_h) + \xi\sigma + \sigma\beta_h + \xi^2\alpha_h = 0
\end{aligned}$$

and

$$\begin{aligned}
q(0,0) &= A((T-T_h)[D_uG(u_h)z_0 - f - G(u_h)], \psi_h) + a_1(z_0; z_0, \psi_h), \\
\frac{\partial q}{\partial \sigma}(0,0) &= \beta_h + a(z_0; z_1, \psi_h) + A((T-T_h)D_uG(u_h)z_1, \psi_h), \\
\frac{\partial q}{\partial \xi}(0,0) &= a(\varphi_h; z_0, \psi_h) + a(z_0; z_2, \psi_h) - A((T-T_h)D_uG(u_h)(z_2-\varphi_h), \psi_h), \\
\frac{\partial^2 q}{\partial \sigma^2}(0,0) &= A((T-T_h)D_uG(u_h)z_{11}, \psi_h) + a(z_0; z_{11}, \psi_h) + a_1(z_1; z_1, \psi_h), \\
\frac{\partial^2 q}{\partial \xi^2}(0,0) &= A((T-T_h)D_uG(u_h)z_{22}, \psi_h) + a(\varphi_h; z_2, \psi_h) \\
&\quad + a_1(z_2; z_2, \psi_h) + a(z_0; z_{22}, \psi_h) + \alpha_h. \\
\frac{\partial^2 q}{\partial \xi \partial \sigma}(0,0) &= A((T-T_h)D_uG(u_u)z_{12}, \psi_h) + a(\varphi_h; z_1, \psi_h) \\
&\quad + a(z_1; z_2, \psi_h) + a(z_0; z_{12}, \psi_h) + 1.
\end{aligned}$$

Utilizing the inequalities

$$|\psi_h|_1 \le \lambda_h^{-1}, \ |u_h|_1 \le \lambda_h^{-1}\|f\|*$$

and (4.29), (4.30), we obtain

$$\begin{aligned}
|q(0,0)| &\le C_1h^{2m} + C_2h^m, \\
\left|\frac{\partial q}{\partial \sigma}(0,0)\right| &\ge |\beta_h| - C_3|z_1|_1h^m, \ \left|\frac{\partial q}{\partial \xi}(0,0)\right| \le C_5h^{2m} + C_6h^m.
\end{aligned}$$

Hence, for $h \to 0$, we have

$$q(0,0) \to 0, \frac{\partial q}{\partial \sigma}(0,0) \to \beta \neq 0 \text{ and } \frac{\partial q}{\partial \xi}(0,0) \to 0.$$

Concluding above, we know that (λ_h, u_h) tends to (λ^*, u^*) which is a simple limit point of (4.1) (cf. Brezzi/Rappaz/Raviart [2]). Furthermore,

$$0 \neq \alpha_h = a(\varphi_h; \varphi_h, \psi_h) \rightarrow a_1(\varphi; \varphi, \psi),$$

where φ and ψ are nullvectors of $\lambda^* I - TDG(u^*)$ and its adjoint operator $\lambda^* I - [TDG(u^*)]^*$ respectively. Therefore, (λ^*, u^*) is a nondegenerate turning point of the stationary Navier-Stokes equations (4.1).

Theorem 4.1 *Let (4.13) and the following conditions 1)-2) be satisfied:*

1) *λ_h is a simple eigenvalue of (4.19) and φ_h is the corresponding eigenvector. For $X_2 := \text{span}\,[\varphi_h], X_1 := X_2^{\perp}, X = X_1 + X_2$, there is a constant $d > 0$, such that*

$$\sup_{v \in X_1} |\lambda_h A(w, v) + a(u_h; w, v)|/|v|_1 \geq d \cdot |w|_1, \forall w \in X_1;$$

2) *There are constants $C_0, \delta \in [0, 1/5]$ independent of $h > 0$, such that*

$$d \cdot \min\{1, |\alpha_h|, |\beta_h|\} \geq C_0 \cdot h^{\delta},$$

where $\alpha_h; = a(\varphi_h; \varphi_h, \psi_h), \beta_h := A(u_h, \psi_h)$. Denote $\Delta\lambda_h := C_0|\alpha_h \beta_h^{-1}|h^{3\delta}$. Then, if sign$(\alpha_h$

$\beta_h) = -1$, there is a constant $h_0 > 0$, such that for all $h \in (0, h_0)$, there exists $\lambda^* \in [\lambda_h - \Delta\lambda_h, \lambda_h + \Delta\lambda_h]$ and

a) *For each $\lambda \in [\lambda_h - \Delta\lambda_h, \lambda^*]$, equation (4.4) has exactly two continuous solution $u^-(\lambda), u^+(\lambda)$;*

b) *For $\lambda = \lambda^*$, equation (4.4) has a unique solution u^* and*

$$\lim_{\lambda \to \lambda^*} u^-(\lambda) = \lim_{\lambda \to \lambda^*} u^+(\lambda) = u^*;$$

c) *For all $\lambda \in (\lambda^*, \lambda^* + \Delta\lambda_h]$, equation (4.4) has no solutions in the neighborhood of (λ_h, u_h).*

Remark 4.1 Similar conclusions hold for sign$(\alpha\beta) = 1$.

Since equation (4.4) has at least one solution for all $\lambda \in \mathbf{R}^+$, the concluion c) above implies that its solution is in fact far away from the point (λ_h, u_h) for λ in $(\lambda^*, \lambda^* + \Delta\lambda_h]$.

Finally, we mention that the verifications of the condition (4.13) is evident and of the conditions 1)-2) are followed by an analysis of the rank deficiency of the matrix $\lambda_h A(\cdot, \cdot) + a(u_h; \cdot, \cdot)$ during the numerical approximations (cf. Chan [3]).

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非线性问题转向点的数值予测

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摘　要

本文讨论了一类非线性问题及其有限元逼近问题的转向点之数值予测, 以及应用于定常的Navier-Stokes问题.