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Solution to the Modified Anisotropic Heisenberg Spin Chain*

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Abstract In this paper, we deal with the global existence and uniqueness of smooth solution to the modified anisotropic Heisenberg spin chain.

§1. Introduction

In the recent years, there has been a tremendous amount of papers coincerning the solvability to the system of Heisenberg spin chain [1-3] and the system with Gilbert damping [4-6], which are important equations occurring in the domain of solid state physics. From the physical view, an interesting problem for certain integrable equations is that some of them do possess a similar but slightly different integrable form known as the modified form. In [7], Archan, K.De, et. al., proposed a modified equation for the system of Heisenberg spin chain, i.e.

$$Z_t = -Z \times Z_{xx} - \frac{1}{2}\{Z(Z, BZ)\}_x + (\alpha + B)Z_x, \tag{1.1}$$

where Z is a three-vector $\{Z_1(x,t), Z_2(x,t), Z_3(x,t)\}$ coupled by the constraint on initial data (magnetically saturated condition):

$$Z_1^2(x,0) + Z_2^2(x,0) + Z_3^2(x,0) = 1, \quad \text{for } x \in R^1, \tag{1.2}$$

α is an arbitrary constant, $B = \mathrm{diag}(b_1, b_2, b_3)$ and $b_i (i = 1,2,3)$ is constant, (a, b) denotes the scalar product and $(a \times b)$ denotes the cross product in R^3. One can easily see that the modified system (1.1) becomes usual Heisenberg spin chain if $\alpha = 0, B = 0$.

In this paper, we study the existence and uniqueness of smooth solution for the system (1.1) with the following initial-boundary conditions:

$$\begin{cases} Z(x-D,t) = Z(x+D,t), & \text{for } t > 0, x \in R^1. \\ Z(x,0) = Z_0(x), & \text{for } x \in R^1, \end{cases} \tag{1.3}$$

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where $D > 0$ is a constant, the initial data $Z_0(x)$ satisfies the magnetically saturated condition (1.2). For the usual system of Heisenberg spin chain, one can establish maximum bound and gradient estimate in L^2 without any difficulty [3,4]. However, we shall find in the sequel, that it is much more diffecult to establish the correspondent estimates for the modified case (1.1).

For simplicity here e.g., we shall omit some well-known procedures , the local existence of smooth solution for the initial problem (1.1) (1.3), and some similar and standard calculations and deductions, there are refered to [3] [4].

By $\tilde{H}^m(\Omega)$, we denote a Sobolev space with norm:

$$\|u\|_{\tilde{H}^m(\Omega)} = \|u\|_\infty + \sum_{k=1}^{m} \|\frac{\partial^k u}{\partial x^k}\|_2 < \infty, \quad \text{for } u \in \tilde{H}^m(\Omega),$$

where $\Omega = (-D, D), m \geq 1$ is an integer, $\|\cdot\|_p (1 \leq p \leq \infty)$ denotes the usual norm $\|\cdot\|_{L^p(\Omega)}$.

Now, we state the result of this paper

Theorem 1.1 *Let $\alpha, b_i (i = 1,2,3)$ be given constants, and let $Z_0(x) \in \tilde{H}^m(R)$ satisfy the condition (1.2), $(m \geq 3)$. If $D > 0, T > 0$ and j, s denote nonnegative integers, then for the problem (1.1) (1.3) there exists a smooth solution $Z(x,t)$, such that*

(a) $|Z(x,t)| = 1$, for $x \in \Omega, t \geq 0$, and $\|D_t^j D_x^s Z(\cdot,t)\|_2 \leq$ const, for $t \in [0,T]$ with $1 \leq 2j + s \leq m$, and the constant depends only on D, m, T, α and B;

(b) if $Z(x,t)$ and $W(x,t)$ are smooth solutions of (1.1) (1.3) with the same initial data, then $Z = W$.

§2. A Priori Estimates

Throughout this section, we assume that the initial data $Z_0(x)$ satisfies the hypotheses of Theorem 1.1. D and T denotes arbitrary positive numbers.

lemma 2.1 *Let $\Omega = (-D, D)$, and $Z(x,t) \in L^\infty(0,T; \tilde{H}^m(\Omega))$. Then we have*

$$|Z(x,t)| = 1, \quad \text{for } x \in \Omega, t \geq 0. \tag{2.1}$$

Proof We multiply (1.1) by $Z(x,t)$ and integrate over Ω, we get

$$\frac{1}{2}(|Z|^2)_t = -\frac{1}{4}(|Z|^2)_x (Z, BZ) - \frac{1}{2}|Z|^2 (Z, BZ)_x + \frac{\alpha}{2}(|Z|^2)_x + \frac{1}{2}(Z, BZ)_x.$$

Set $V = |Z|^2 - 1$. The above identity may be rewritten as

$$V_t = -\frac{1}{2} V_x (Z, BZ) - V(Z, BZ)_x + \alpha V_x. \tag{2.2}$$

From (1.3) and condition (1.2), we see that

$$V(x-D,t)=V(x+D,t), V(x,0)=0, \quad \text{for } x\in\Omega, t\geq 0. \tag{2.3}$$

After integration by parts, (2.2) (2.3) gives

$$\frac{d}{dt}\|V(\cdot,t)\|_2^2=-\frac{3}{2}\infty_\Omega V^2(Z,BZ)_x dx\leq\frac{3}{2}\|(Z,BZ)_x\|_\infty\|V(\cdot,t)\|_2^2.$$

Thus, by using Gronwall inequality, (2.1) follows immediately from the above inequality.

Lemma 2.2 *Let $Z(x,t)\in L^\infty(0,T;\tilde{H}^m(\Omega))$, then we have*

$$\begin{aligned}&\frac{d}{dt}\{\int_\Omega |Z_x|^2dx+\int_\Omega Z_x\cdot(Z\times BZ)dx\}\\ &\quad=-\int_\Omega\{(Z\times BZ_x)+2(Z_x\times BZ)+B(Z\times Z_x)\}\cdot f_x(Z),\end{aligned} \tag{2.4}$$

where $f(Z)=-\frac{1}{2}Z(Z,BZ)+(\alpha+B)Z$.

Proof If we multiply (1.1) by Z_{xx} and integrate over Ω, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega|Z_x|^2dx&=\int_\Omega Z_{xx}\cdot\{\frac{1}{2}[Z(Z,BZ)]_x-(\alpha+B)Z_x\}dx\\ &=-\frac{3}{4}\int_\Omega|Z_x|^2(Z,BZ)_x dx,\end{aligned} \tag{2.5}$$

where we have used the integration by parts and the identities:

$$Z\cdot Z_x=0,\quad Z\cdot Z_{xx}=-|Z_x|^2. \tag{2.6}$$

On the other hand, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega Z_x\cdot(Z\times BZ)dx&=-\int_\Omega(Z\times Z_x)\cdot BZ_t-2\int_\Omega(Z_x\times BZ)\cdot Z_t\\ &\quad-\int_\Omega(Z\times BZ_x)\cdot Z_t=I_1+I_2+I_3,\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}I_1&=\int_\Omega(Z\times Z_x)\cdot B(Z\times Z_x)_x-\int_\Omega(Z\times Z_x)\cdot Bf_x(Z)\\ &=-\int_\Omega B(Z\times Z_x)\cdot f_x(Z),\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}I_2&=2\int_\Omega(Z_x\times BZ)\cdot(Z\times Z_{xx})-2\int_\Omega(Z_x\times BZ)\cdot f_x(Z)\\ &=2\int_\Omega\{(Z\cdot Z_x)(Z_{xx}\cdot BZ)-(Z_x\cdot Z_{xx})(Z\cdot BZ)\}-2\int_\Omega(Z_x\times BZ)\cdot f_x(Z)\end{aligned}$$

$$= \int_\Omega |Z_x|^2 (Z, BZ)_x dx - 2\int_\Omega (Z_x \times BZ)\cdot f_x(Z) dx, \tag{2.9}$$

and the third term

$$\begin{aligned} I_3 &= \int_\Omega (Z \times BZ_x)\cdot(Z\times Z_{xx}) - \int_\Omega (Z\times BZ_x)\cdot f_x(Z) \\ &= \int_\Omega \{|Z|^2 Z_{xx}\cdot BZ_x - (Z\cdot Z_{xx})\cdot(Z\cdot BZ_x)\} - \int_\Omega (Z\times BZ_x)\cdot f_x(Z) \\ &= \frac{1}{2}\int_\Omega |Z_x|^2 (Z,BZ)_x dx - \int_\Omega (Z\times BZ_x)\cdot f_x(Z) dx. \end{aligned} \tag{2.10}$$

In the above identities, we have used the facts:

$$\begin{aligned} A_1 \times (A_2\times A_3) &= (A_1\cdot A_3)A_2 - (A_1\cdot A_2)A_3, \\ (A_1\times A_2)\cdot(A_3\times A_4) &= (A_1\cdot A_3)(A_2\cdot A_4) - (A_1\cdot A_4)(A_2\cdot A_3), \end{aligned}$$

and

$$(A_{1x}, BA_1) = \frac{1}{2}(A_1, BA_1)_x,$$

for $A_i \in R^3, i = 1,2,3,4$. Hence, (2.7) gives

$$\begin{aligned} \frac{d}{dt}\int_\Omega Z_x\cdot(Z\times BZ)dx = \frac{3}{2}\int_\Omega |Z_x|^2(Z,BZ)_x dx - \int_\Omega \{B(Z\times Z_x) + 2(Z_x\times BZ) \\ +(Z\times BZ_x)\}\cdot f_x(Z) dx. \end{aligned} \tag{2.11}$$

Combining (2.11) with (2.5), we obtain the desired identity (2.4).

Corollary 2.1 *We have*

$$\|Z_x(\cdot,t)\|_2 \le const, \tag{2.12}$$

where the constant depends only on D, T, α, B and the norm $\|Z_{0x}\|_2$.

Proof By virtue of Lemma 2.1, the estimate (2.12) can be easily deduced from the identity (2.4).

Lemma 2.3 *Let $Z(x,t) \in L^\infty(0,T;\tilde{H}^m(\Omega))$, then we have*

$$\begin{aligned} &\frac{d}{dt}\{\int_\Omega |Z_{xx}|^2 dx - \frac{5}{4}\int_\Omega |Z_x|^4 dx + \frac{5}{3}\int_\Omega Z_{xx}\cdot(Z_x\times BZ)dx \\ &\qquad -\frac{10}{3}\int_\Omega Z_{xx}\cdot(Z\times BZ_x)dx - \frac{5}{6}\int_\Omega (Z,BZ)(Z\times Z_{xx})\cdot Z_x dx\} \\ &= R(t), \end{aligned} \tag{2.13}$$

where $R(t)$ is such that

$$|R(t)| \le const\{\|Z_{xx}(\cdot,t)\|_2^2 + \|Z_x(\cdot,t)\|_6^6 + 1\},$$

for $t \ge 0$, and the constant depends only on Ω, α, and B.

Proof Notice that the result of Lemma 2.1, we see that

$$Z \cdot Z_x = 0, Z \cdot Z_{xx} = -|Z_x|^2, Z \cdot Z_{xxx} = -3Z_x \cdot Z_{xx}. \tag{2.14}$$

If $(x,t) \in S = \{x \in \Omega, t \in [0,T]; |Z_x(x,t)| \neq 0\}$, then by use of the orthogonality of the three vectors Z, Z_x and $Z \times Z_x$, we have

$$Z_{xx} = \alpha Z + \beta Z_x + \gamma Z \times Z_x, \tag{2.15}$$

where

$$\alpha = -|Z_x|^2, \beta = Z_x \cdot Z_{xx}/|Z_x|^2, \gamma = (Z \times Z_x) \cdot Z_{xx}/|Z_x|^2.$$

In what follows, our intention is to proceed by checking the terms in the left hand side of (2.13) in several steps.

Step 1 Applying the identities in (2.14), and integration by parts, we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega |Z_{xx}|^2 dx &= 2\int_\Omega Z_{xx} \cdot (-Z \times Z_{xx} - \frac{1}{2}\{Z(Z,BZ)\}_x + (\alpha + B)Z_x)_{xx} \\ &= 2\int_\Omega Z_{xxx} \cdot (Z_x \times Z_{xx}) dx - \int_\Omega Z_{xx} \cdot \{Z(Z,BZ)\}_{xxx} dx, \end{aligned} \tag{2.16}$$

where, by use of (2.15) and (2.14)

$$\begin{aligned} & 2\int_\Omega Z_{xxx} \cdot (Z_x \times Z_{xx}) dx \\ & = 2\int_\Omega Z_{xxx} \cdot \{Z_x \times (\alpha Z + \beta Z_x + \gamma Z \times Z_x)\} dx \\ & = 2\int_\Omega |Z_x|^2 (Z \times Z_x) \cdot Z_{xxx} dx + 2\int_\Omega (Z \times Z_x) \cdot Z_{xx}(Z \cdot Z_{xxx}) dx \\ & = -2\int_\Omega |Z_x|^2 Z_x \cdot (Z \times Z_{xx})_x dx - 2\int_\Omega (Z_x \cdot (Z \times Z_{xx})(-\frac{3}{2}|Z_x|^2)_x dx \\ & = -5\int_\Omega |Z_x|^2 Z_x \cdot (Z \times Z_{xx})_x dx \\ & = 5\int_\Omega |Z_x|^2 Z_x \cdot (Z_t - f_x(Z))_x dx \\ & = \frac{5}{4}\frac{d}{dt}\int_\Omega |Z_x|^4 dx - 5\int_\Omega |Z_x|^2 Z_x \cdot f_{xx}(Z) dx, \end{aligned}$$

and

$$\begin{aligned} & -\int_\Omega Z_{xx} \cdot \{Z(Z,BZ)\}_{xxx} dx \\ & = -\int_\Omega Z_{xx} \cdot \{Z_{xxx}(Z,BZ) + 3Z_{xx}(Z,BZ)_x + 3Z_x(Z,BZ)_{xx} + Z(Z,BZ)_{xxx}\} \\ & = -\frac{5}{2}|Z_{xx}|^2 (Z,BZ)_x dx + \frac{5}{2}\int_\Omega |Z_x|^2 (Z,BZ)_{xxx} dx. \end{aligned}$$

Therefore, (2.16) gives

$$\frac{d}{dt}\int_\Omega |Z_{xx}|^2dx - \frac{5}{4}\frac{d}{dt}\int_\Omega |Z_x|^4dx = -\frac{5}{2}\int_\Omega |Z_{xx}|^2(Z,BZ)_xdx$$

$$+\frac{5}{2}\int_\Omega |Z_x|^2(Z,BZ)_{xxx}dx - 5\int_\Omega |Z_x|^2 Z_x\cdot f_{xx}(Z)dx, \tag{2.17}$$

where $f(Z) = -\frac{1}{2}Z(Z,BZ) + (\alpha + B)Z$.

Step 2

$$\frac{d}{dt}\int_\Omega Z_{xx}\cdot(Z_x\times BZ)dx = -2\int_\Omega Z_{xt}\cdot(Z_{xx}\times BZ)dx$$

$$-\int_\Omega Z_{xt}\cdot(Z_x\times BZ_x)dx + \int_\Omega (Z_{xx}\times Z_x)\cdot BZ_tdx, \tag{2.18}$$

where

$$\begin{aligned}
&-2\int_\Omega Z_{xt}\cdot(Z_{xx}\times BZ)dx\\
&= -2\int_\Omega\{-Z\times Z_{xx} + f_x(Z)\}_x\cdot(Z_{xx}\times BZ)dx\\
&= 2\int_\Omega (Z\times Z_{xxx})\cdot(Z_{xx}\times BZ) + 2\int_\Omega (Z_x\times Z_{xx})\cdot(Z_{xx}\times BZ)\\
&\quad -2\int_\Omega f_{xx}(Z)\cdot(Z_{xx}\times BZ)\\
&= 2\int_\Omega (Z_{xx},BZ)(Z_x\cdot Z_{xx}) + 2\int_\Omega (Z_{xxx},BZ)(Z\cdot Z_{xx})\\
&\quad -2\int_\Omega f_{xx}(Z)\cdot(Z_{xx}\times BZ)\\
&= -\frac{3}{2}\int_\Omega |Z_x|^2(Z,BZ)_{xxx}dx - 8\int_\Omega (Z_x,BZ_x)(Z_x\cdot Z_{xx})dx\\
&\quad -2\int_\Omega f_{xx}\cdot(Z_{xx}\times BZ)dx,
\end{aligned} \tag{2.19}$$

the second term in the right hand side of (2.18)

$$\begin{aligned}
-\int_\Omega Z_{xt}\cdot(Z_x\times BZ_x)dx &= -\int_\Omega\{-Z\times Z_{xx} + f(Z)\}_x\cdot(Z_x\times BZ_x)dx\\
&= \int_\Omega (Z\times Z_{xxx})\cdot(Z_x\times BZ_x)dx + \int_\Omega (Z_x\times Z_{xx})\cdot(Z_x\times BZ_x)dx\\
&\quad -\int_\Omega f_{xx}(Z)\cdot(Z_x\times BZ_x)dx
\end{aligned}$$

$$= \frac{1}{2}\int_\Omega |Z_{xx}|^2(Z,BZ)_x dx - \frac{1}{4}\int_\Omega |Z_x|^2(Z,BZ)_{xxx}dx$$

$$+ \int_\Omega |Z_x|^2(Z_x, BZ_x)_x dx - \int_\Omega f_{xx}(Z)\cdot(Z_x\times BZ_x)dx, \tag{2.20}$$

and the last term in the right hand side of (2.18)

$$\int_\Omega (Z_{xx}\times Z_x)\cdot BZ_t dx$$

$$= \int_\Omega (Z_{xx}\times Z_x)\cdot B(-Z\times Z_{xx} + f_x(Z))dx$$

$$= -\int_\Omega \{Z_x\times(\alpha Z + \beta Z_x + \gamma Z\times Z_x)\}\cdot B(Z\times Z_{xx})dx$$

$$+ \int_\Omega (Z_{xx}\times Z_x)\cdot Bf_x(Z)dx$$

$$= \int_\Omega (Z\times Z_x)\cdot Z_{xx}\{Z\cdot B(Z\times Z_{xx})\}dx + \int_\Omega |Z_x|^2(Z\times Z_x)\cdot B(Z\times Z_{xx})dx$$

$$+ \int_\Omega (Z_{xx}\times Z_x)\cdot Bf_x(Z)dx. \tag{2.21}$$

Combining (2.19)-(2.21) with (2.18), we obtain

$$\frac{d}{dt}\int_\Omega Z_{xx}\cdot(Z_x\times BZ)dx$$

$$= \frac{1}{2}\int_\Omega |Z_{xx}|^2(Z,BZ)_x dx - \frac{7}{4}\int_\Omega |Z_x|^2(Z,BZ)_{xxx}dx$$

$$+ \int_\Omega (Z\times Z_x)\cdot Z_{xx}\{Z\cdot B(Z\times Z_{xx})\}dx + 5\int_\Omega |Z_x|^2(Z_x, BZ_x)_x dx$$

$$+ \int_\Omega |Z_x|^2(Z\times Z_x)\cdot B(Z\times Z_{xx})dx - 2\int_\Omega f_{xx}\cdot(Z_{xx}\times BZ)dx$$

$$- \int_\Omega f_{xx}\cdot(Z_x\times BZ_x)dx + \int_\Omega (Z_{xx}\times Z_x)\cdot Bf_x dx. \tag{2.22}$$

Step 3 Repeating some similar calculations as for (2.16) (2.17) in Step 1 and Step 2, we can check that

$$\frac{d}{dt}\int_\Omega Z_{xx}\cdot(Z\times BZ_x)dx$$

$$= -\int_\Omega |Z_x|^2(Z,BZ)_{xxx}dx + 5\int_\Omega |Z_x|^2 Z_{xx}\cdot BZ_x dx$$

$$+ \int_\Omega Z_{xx}\cdot\{f_x(Z)\times BZ_x\}dx - \int_\Omega f_{xx}(Z)\cdot(Z\times BZ_x)_x dx$$

$$- \int_\Omega f_{xx}(Z)\cdot B(Z\times Z_{xx})dx, \tag{2.23}$$

and

$$\frac{d}{dt}\int_\Omega (Z,BZ)\{(Z\times Z_{xx})\cdot Z_x\}dx$$
$$=2\int_\Omega |Z_{xx}|^2(Z,BZ)_x dx-\frac{1}{2}\int_\Omega |Z_x|^2(Z,BZ)_{xxx}dx$$
$$+2\int_\Omega (Z\times Z_x)\cdot Z_{xx}\{Z\cdot B(Z\times Z_{xx})\}dx$$
$$+\frac{3}{2}\int_\Omega |Z_x|^4(Z,BZ)_x dx+2\int_\Omega f_x(Z)\cdot BZ\{(Z\times Z_{xx})\cdot Z_x\}dx$$
$$+\int_\Omega (Z,BZ)(Z\times Z_{xx})\cdot f_{xx}(Z)dx+\int_\Omega (Z,BZ)\{f_x(Z)\times Z_{xx}\}dx$$
$$-\int_\Omega f_{xx}(Z)\cdot\{(Z,BZ)Z_x\}_x dx. \tag{2.24}$$

Finally, combining with the four identities (2.17) (2.22) (2.23) and (2.24), eliminating the terms: $\int_\Omega |Z_{xx}|^2(Z,BZ)_x dx$, $\int_\Omega |Z_x|^2(Z,BZ)_{xxx}dx$ and $\int_\Omega (Z\times Z_x)\cdot Z_{xx}\{Z\cdot B(Z\times Z_{xx})\}dx$ in their right hand side, we then obtain the desired identity (2.13).

Corollary 2.2 *We have*

$$\|Z_{xx}(\cdot,t)\|_2\le \ const,\ \ for\ t\in[0,T], \tag{2.25}$$

where the constant depends only on α, B, D, T and the norm $\|Z_{0xx}\|_2$.

Proof From the identity (2.13), by Hölder inequality and the following interpolation inequalities

$$\|Z_x\|_4\le C(\|Z_{xx}\|_2+\|Z_x\|_2)^{\frac{1}{4}}\|Z_x\|_2^{\frac{3}{4}},\ \|Z_x\|_6\le C(\|Z_{xx}\|_2+\|Z_x\|_2)^{\frac{1}{3}}\|Z_x\|_2^{\frac{2}{3}},$$

we can easily obtain the estimate (2.25).

Lemma 2.4 *Let $Z(x,t)\in L^\infty(0,T;\tilde{H}^m(\Omega))$. Then we have*

$$\|D_x^m Z(\cdot,t)\|_2\le \ const,\ \ for\ t\in[0,T], \tag{2.27}$$

where the constant depends only on α, B, D, T and the norm $\|D_x^m Z_0\|_2$.

Proof We shall verify the estimate (2.27) by induction. Assume that

$$\|D_x^j Z(\cdot,t)\|_2\le \ \text{const, for } t\in[0,T] \tag{2.28}$$

with $2\le j\le k$, where the conatant is allowed to depend on the norm $\|D_x^j Z_0\|_2$. We shall then prove that (2.28) holds for $j=k+1$.

If we differentiate equation (1.1) $(k+1)$-times with respect to x, and multiply the result equation by $D_x^{k+1}Z(x,t)$, then

$$\frac{d}{dt}\|D_x^{k+1}Z(\cdot,t)\|_2^2=2\int_\Omega D_x^{k+2}Z\cdot D_x^k(Z\times Z_{xx})dx+2\int_\Omega D_x^{k+1}Z\cdot D_x^{k+2}f(Z)dx. \tag{2.29}$$

The second term on the right of (2.29) may be bounded as follows:

$$
\begin{aligned}
&2\int_\Omega D_x^{k+1}Z\cdot D_x^{k+2}f(Z)dx \\
&\quad = 2\int_\Omega D_x^{k+1}Z\cdot D_x^{k+2}\{-\frac{1}{2}Z(Z,BZ)+(\alpha+B)Z\}dx \\
&\quad = -\sum_{j=0}^{k+2}\int_\Omega D_x^{k+1}Z\cdot D_x^jZD_x^{k+2-j}(Z,BZ)dx \\
&\quad = -\int_\Omega (D_x^{k+1}Z\cdot Z)D_x^{k+2}(Z,BZ)dx \\
&\qquad -(k+2)\int_\Omega (D_x^{k+1}Z\cdot Z_x)D_x^{k+1}(Z,BZ)dx \\
&\qquad -\int_\Omega (D_x^{k+1}Z\cdot D_x^{k+2}Z)(Z,BZ)dx \\
&\qquad -(k+2)\int_\Omega (D_x^{k+1}Z\cdot D_x^{k+1}Z)(Z,BZ)_xdx \\
&\qquad -\sum_{j=2}^{k}C_{k+2}^j\int_\Omega (D_x^{k+2}Z\cdot D_x^jZ)D_x^{k+2-j}(Z,BZ)dx \\
&\quad = -\sum_{j=1}^{k+1}C_{k+1}^j\int_\Omega (D_x^jZ\cdot D_x^{k+2-j}Z)D_x^{k+1}(Z,BZ)dx \\
&\qquad -(k+1)\int_\Omega (D_x^{k+1}Z\cdot Z_x)D_x^{k+1}(Z,BZ) \\
&\qquad -(k+\frac{3}{2})\int_\Omega |D_x^{k+1}Z|^2(Z,BZ)_xdx \\
&\qquad -\sum_{j=2}^{k}C_{k+2}^j\int_\Omega (D_x^{k+1}Z\cdot D_x^jZ)D_x^{k+2-j}(Z,BZ)dx \\
&\quad \le C\{1+\|D_x^{k+1}Z(\cdot,t)\|_2^2\}, \qquad (2.30)
\end{aligned}
$$

where we have used Corollary 2.2, the interpolation inequality:

$$\|D_x^kZ\|_\infty \le C(\|D_x^{k+1}Z\|_2+\|D_x^kZ\|_2)^{\frac{1}{2}}\|D_x^kZ\|_2^{\frac{1}{2}},$$

and the following identity:

$$Z\cdot D_x^{k+2}Z=-\sum_{j=1}^{k+1}C_{k+1}^jD_x^jZ\cdot D_x^{k+2-j}Z, \quad (\text{since } Z\cdot Z_x=0).$$

We now consider the first term on the right hand side of (2.29)

$$2\int_\Omega D_x^{k+2}Z\cdot D_x^k(Z\times Z_{xx})dx$$

$$=2\sum_{j=0}^{k}C_k^j\int_\Omega D_x^{k+2}Z\cdot\{(D_x^jZ)\times(D_x^{k+2-j}Z)\}dx$$

$$=2\int_\Omega D_x^{k+2}Z\cdot\{Z_x\times D_x^{k+1}Z\}dx$$

$$+2\sum_{j=2}^{k}C_k^j\int_\Omega D_x^{k+2}Z\cdot\{(D_x^jZ)\times(D_x^{k+2-j}Z\}dx, \tag{2.31}$$

where the second term on the right hand side can be bounded as in (2.30), namely, we have

$$2\sum_{j=2}^{k}C_k^j\int_\Omega D_x^{k+2}Z\cdot\{(D_x^jZ)\times(D_x^{k+2-j}Z)\}dx\le C\{1+\|D_x^{k+1}Z(\cdot,t)\|_2^2\}. \tag{2.32}$$

In order to bound the first term on the right hand side of (2.31), we note that

$$D_x^{k+1}Z=\alpha'Z+\beta'Z_x+\gamma'Z\times Z_x,\ \text{for } (x,t)\in S,$$

where

$$\alpha'=Z\cdot D_x^{k+1}Z=-\sum_{j=1}^{k}C_k^jD_x^jZ\cdot D_x^{k+1-j}Z,\beta'=(Z_x\cdot D_x^{k+1}Z)/|Z_x|^2,$$

and $\gamma'=(Z\times Z_x)\cdot D_x^{k+1}Z/|Z_x|^2$.

We have

$$2\int_\Omega D_x^{k+2}Z\cdot\{Z_x\times D_x^{k+1}Z\}dx$$

$$=2\int_\Omega D_x^{k+2}Z\cdot\{Z_x\times(\alpha'Z+\gamma'Z\times Z_x)\}dx$$

$$=2\int_\Omega D_x^{k+1}Z\cdot\{\alpha'Z\times Z_x\}_xdx+2\int_\Omega\gamma'|Z_x|^2(Z\cdot D_x^{k+2}Z)dx$$

$$=-2\sum_{j=1}^{k}C_k^j\int_\Omega D_x^{k+1}Z\cdot\{(D_x^jZ\cdot D_x^{k+1-j}Z)Z\times Z_x\}_xdx$$

$$-2\sum_{j=1}^{k+1}C_{k+1}^j\int_\Omega(D_x^jZ\cdot D_x^{k+2-j}Z)\{(Z\times Z_x)\cdot D_x^{k+1}Z\}dx$$

$$\le C(1+\|D_x^{k+1}Z\|_2^2). \tag{2.33}$$

Consequently, we combine (2.30)-(2.33) with (2.29), and obtain

$$\frac{d}{dt}\|D_x^{k+1}Z\|_2^2\le\ \text{const}\ (1+\|D_x^{k=1}Z\|_2^2),$$

for $t \in [0,T]$, and Gronwall inequality gives the desired result.

Corollary 2.3 *We have*

$$\|D_t^j D_x^s Z(\cdot,t)\|_2 \leq \ const \ \ for\ t \in [0,T]$$

with $1 \leq 2j+s \leq m$, *here the constant depends only on* α, B, D, T, m *and the norm* $\|D_x^m Z_0\|_2$.

The uniqueness of smooth solution for the problem (1.1) (1.3) can be proved by standard $L^2(\Omega)$-energy estimate, which is omitted.

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修正的各向异性旋转链方程的整体解

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摘 要

本文研究一类修正的海森堡旋转链方程组整体光滑解的存在性和唯一性.