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Multiple Subharmonic Solutions of Nonautonomous Hamiltonian System*

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Abstract By means of Z_p-index and its pseudo-index, we study the existence of multiple subharmonic solutions with prescribed minimal period for nonautonomous superquadratic Hamiltonian system $J\dot{z} = H_z(z(t), t), z(0) = z(2\pi p)$ where $H(z, t + 2\pi p) = H(z, t), t \in R, z \in R^{2N}$. Under hypotheses H1-H4, there are at least kN distinct soultions with minimal period $2\pi p$.

§1. Introduction

In this paper, we establish a multiplicity result of the following nonautonomous superquadratic Hamiltonian system with minimal period $2\pi p$ for any integer $p > 1$, that is

$$(1) \qquad J\dot{z} = H_z(z(t), t), \quad z(0) = z(2\pi p)$$

where $z \in R^{2N}, H(z, t + 2\pi p) = H(z, t), \forall z \in R^{2N}$, with the following hypotheses on H:

(H1) $H(\cdot, t) \in C^1(R^{2N}, R), t \in [0, 2\pi p]$;

(H2) $\exists \beta > 2, \beta H(z, t) \leq (H_z(z, t), z)$;

(H3) $\exists c_2 \geq 0, |H_z(z, t)| \leq c_2 |z|^{\beta - 1}$;

(H4) $H(z, t) \geq c_1 |z|^{\beta}$.

Classically solutions of (1) called subharmonics. The first results in this area were local in nature found using perturbation techniques. Under suitable assumption on H near the origin, there exists a sequence of subharmonics with arbitrary large minimal period. (See [4], [8]) Subsequently new results were obtained by a global approach using calculus of variations. It has been proved that there are distinct solutions $z(p)$; Futhermore, if H has subqradratic of superquadratic grown at the origion and infinity, then the minimal period of $z(p)$ tends to infinity as $p \to \infty$. See [9] and [5]. Recently in [7] a more precise

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result was obtained showing the mutiplicity existence of minimal period for subquadratic nonautonomous system under some "stronger" conditions.

In this paper, we obtain the following theorems of multiplicity Harmonic solutions with the minimal period $2\pi p$ for nonautonomous superquadratic hamiltonian system (1).

Theorem 1.1 *For any integer $p > 1$, such that $c_2/c_1^{\frac{\beta+2}{2\beta}} < \sqrt{2}\beta/\sqrt{k}$, where $1 \le k < s_p, s_p$ is the least prime factor of p. Then under assumptions H1-H4, there are at least kN distinct solutions with minimal perlod $2\pi p$ of system (1).*

Theorem 1.2 *Let H be as in theorem 1.1 with $k = 1$, that is $c_2/c_1^{\frac{\beta+2}{2\beta}} < \sqrt{2}\beta$. Then for any integer $p > 1$, there exist at least N distinct solutions with minimal period $2\pi p$ of system (1).*

For the autonomous cases, it is easily seen that

Corollary 1 *Let $H = H(z) \in C^1(R^{2N}, R)$ sastisfies H1-H4 with $c_2/c_1^{\frac{\beta+2}{2\beta}} < \sqrt{2}\beta$. Then hamiltonian system $J\dot{z} = H'(z)$ has at least N periodic solutions with minimal period $2\pi p$ of system (1).*

The proof of the theorem relys on the theory of Z_p-index i_p and its pseuo-index i_p^* in section 4 which is based on the theory in [2]. Therefore, the difficulty of the indifefinite functional I, which always appear in the study of superquadratic hamiltionian system is overcome. In general, it is diffcult to check the P.S condition and the existence of suitable finite interval of minimax value for the multiplicity solutions.

In this paper, we first transform the problem (1) equivalently to the problem of critical value of an indefinite functional I; then some estimates of I are given in section 2 and section 3 which is used to prove the minimal period of the solution; at last, the proof of Theorem 1.1 is given in section 5.

§2. The Equivalent Problem

In this section, we transform the problem (1) equivalently to the critical value problem of functional I. Set $E = H^{1/2}([0, 2\pi p], R^{2N})$.

For any $w \in E$, define the linear operator $K : E \to E$ as: $\langle v, Kw\rangle = I_0'(w), v, w \in E$ and I_0 is the extension of the functional $\int_0^{2\pi p} wJ\dot{w}dt$ from $C^\infty([0, 2\pi p], R^{2N})$ into $E, \langle\cdot,\cdot\rangle$ is the inner product in Hilbert space E. It can be proved that:

(1) K is a bounded selfadjoint operator,

(2) $\sigma(K) = \{k_j : k_j = \frac{\frac{j}{p}}{1+|\frac{j}{p}|},\ j \in Z\backslash\{0\}\}$.

Let E_j be the eigenspace relative to the eigenvalue k_j, then

$$E_j = \text{span}\{\sin(j/p)te_k + \cos(j/p)te_{k+N},\ \cos(j/p)te_k - \sin(j/p)te_{k+N}\}$$

where $k = 1, \dots, N,\ e_k = (0, \cdots, 1, \cdots, 0) \in r^{2N}$

Under the identification $(u_1, \cdots, u_{2N})$, with $(u_1 + iu_{N+1}, \cdots, u_N + iu_{2N}) \in C^N$, we have $E_j = \{u(t) = \varsigma e^{-i(\frac{j}{p})t}, \varsigma \in C^N\}$.

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Set $E^+ = E_1 \oplus E_2 \oplus \cdots$, $E^- = E_{-1} \oplus E_{-2} \oplus \cdots$. and $E^0 = \mathrm{Ker} K \cong R^{2N}$, then $E = E^+ \oplus E^- \oplus E^0$.

By the principle of variational, the original problem (1) is transformed into the problem of critical points of functional $I : E \to R$

$$I(z) = 1/2\langle z, Kz\rangle - \int_0^{2\pi p} H(z,t)dt, \ z \in E.$$

For $k \in N$, set $H_k = E_1 \oplus E_2 \oplus \cdots \oplus E_k$, $H^+ = E^+$, $H^- = H_k + E^- + E^0$, then we have estimate,

Proposition 2.1 *For any* $z \in H^-, I(z) \le h(\beta) = \pi k^{\frac{\beta}{\beta-2}} (\frac{1}{\beta p c_1})^{\frac{2}{\beta-2}} \frac{\beta-2}{\beta}$

Proof For $z \in H^-$, that is $z = z_k + z^- + z^0$, with $z_k = \sum_{j=1}^{k} \varsigma_j e^{-i(j/p)t} \in H_k$, $z^- \in E^-$ and $z^0 \in E^0$, one has

$$\begin{aligned}
\frac{1}{2}\langle z, Kz\rangle &\le \frac{1}{2} < z_k, Kz_k >= \frac{1}{2}\int_0^{2\pi p} (z_k, J\dot{z}_k > dt \\
&= \frac{1}{2}\int_0^{2\pi p} \sum_{j=1}^{k} (j/p) < \varsigma_j e^{-i(j/p)t}, \varsigma_j e^{-i(j/p)t} > dt \\
&\le \frac{k}{2p}\int_0^{2\pi p} \sum_{j=1}^{k} |\varsigma_j|^2 dt = \frac{k}{2p}\int_0^{2\pi p} |z_k|^2 dt \le \frac{k}{2p}\int_0^{2\pi p} |z|^2 dt \\
&\le \frac{k}{2p}(2\pi p)^{\frac{(\beta-2)}{\beta}} (\int_0^{2\pi p} |z|^\beta dt)^{2/\beta}
\end{aligned}$$

hence by H4,

$$\begin{aligned}
I(z) &\le \frac{1}{2}\langle z, Kz\rangle - \int_0^{2\pi p} c_1 |z|^\beta dt \\
&\le \frac{k}{p}(2\pi p)^{\frac{(\beta-2)}{\beta}} (\int_0^{2\pi p} |z|^\beta dt)^{2/\beta} - c_1 \int_0^{2\pi p} |z|^\beta dt \\
&\le \pi k^{\frac{(\beta-2)}{\beta}} (\frac{1}{\beta p c_1})^{\frac{2}{\beta-2}} \frac{\beta-2}{\beta}.
\end{aligned}$$

§3. The Estimate of Minimal Period Solution

In this section, we give a sufficient condition for the minimal perlod of solution which will be used in section 5.

For any solution of system (1) in E, we have following estimates.

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Proposition 3.1 *Suppose H sastisfies H1-H4, z is a solution of system (1) with minimal period $2\pi p/m, m \geq 2$, then*

$$I(z) \geq g(\beta) = (\frac{2\beta c_1}{c_2^2})^{\frac{\beta}{\beta-2}}(\beta-2)\pi p^{\frac{-2}{\beta-2}}.$$

Proof By Wrintinger's inequality, $||u||_{L^2} \leq \frac{T}{2\pi}||\dot{u}||_{L^2}$, in which T is the period of $u(t)$, we have

$$||z||_{L^2} \leq (1/m)(2\pi p/2\pi)||\dot{z}||_{L^2} \leq (p/m)||\dot{z}||_{L^2}.$$

By H2-H4 and Hölder's inequality,

$$\begin{aligned}
\beta\int_0^{2\pi p} H(z(t),t)dt &\leq \int_0^{2\pi p}\langle H'(z,t),z\rangle dt = \int_0^{2\pi p}\langle z, J\dot{z}\rangle dt \\
&\leq ||J\dot{z}||_{L^2}^2 = \frac{p}{m}||H_z(z,t)||_{L^2}^2 \leq \frac{p}{m}c_2^2\int_0^{2\pi p}|z(t)|^{2\beta-2}dt \\
&\leq \frac{p}{m}c_2^2\frac{M}{c_1}\int_0^{2\pi p} H(z,t)dt
\end{aligned}$$

in which $M = \max\{|z(t)|^{\beta-2}, t \in [0, 2\pi p]\}$. By H4, $H(z,t) > 0$ for $z \neq 0$, so $M \geq \frac{m\beta c_1}{pc_2^2}$ and

$$H(z,t) \geq M^{\frac{\beta}{\beta-2}} \geq (m\beta c_1/pc_2^2)^{\frac{\beta}{\beta-2}}$$

Moreover, according to H2,

$$\begin{aligned}
I(z) &= \int_0^{2\pi p}[\frac{1}{2}\langle z, J\dot{z}\rangle - H(z,t)]dt = \int_0^{2\pi p}[\frac{1}{2}(z, H_z(z,t) - H(z,t)]dt \\
&\geq \frac{\beta-2}{2}\int_0^{2\pi p} H(z,t)dt \geq \pi(\beta-2)(m\beta c_1/c_2^2)^{\frac{\beta}{\beta-2}}p^{\frac{-2}{\beta-2}} \\
&\geq g(\beta).
\end{aligned}$$

Comparing with proposition 2.1, we have the following estimate of the minimal periodic solution.

Proposition 3.2 *Suppose H sastisfies condition H1-H4 and that for c_1, c_2, if $c_2/c_1^{\frac{\beta+2}{2\beta}} < \sqrt{2}\beta/\sqrt{k}$, then for any periodic solution $z \in H^-$ of system (1), $2\pi p$ is its minimal period.*

Proof By Proposition 2.1, for any $z \in H^-$ we have $I(z) \leq h(\beta)$. Meanwhile, as $c_2/c_1\frac{\beta+2}{2\beta} \leq \sqrt{2}\beta/\sqrt{k}$, it is easy check that $h(\beta) < g(\beta)$, for $\beta > 2$, so that $I(z) < g(\beta)$, which is contradiction to Proposition 3.1. So we have to take $m = 1$, solution $z(t)$ of system (1) has minimal period $2\pi p$.

§4. Theory of Z_p-index and Its Pseudoindex

As we know that functional I is an indefinite functional which is neither bounded from below nor from above. In order to overcome this obstacle, we have to use the theory of Z_p index and its pseudoindex on Banach space E which is based on [2].

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Define the norm-preserving operator:

$$T : E \to E; \quad Tu = u(t + 2\pi), \ T^p = id$$

then, T generates a Z_p group action on E.

Definition 4.1 *The subset A of E is called invariant if $TA \subset A$.*

Definition 4.2 *The continuous map $f : E \to E$ is called T-equivariant if $f(Tu) = Tf(u), u \in E$.*

Set $\sum = \{A \subset E : A$ is closed and invariant$\}$, $W = \{h \in C(E, E), h$ is T-equivariant$\}$.

Define the index map $i_p : \sum \to N \cup \{+\infty\}$, $N = \{n \in Z | n > 0\}$ as follows:

For any $A \in \sum, i_p(A) = k$, in which k is the smallest nonnegative integer such that there exists a continuous map $h : A \to C^k \backslash \{0\}, h = (h_1, \cdots, h_k)$ and integers $m_j \neq 0$ is relatively prime to $p, 1 \leq j \leq k$ such that $h_j(Tu) = e^{imj(2\pi/p)} h_j(m)$. Set $i_p(A) = 0$ if $A = \phi$ and $i_p(A) = +\infty$, if no such map exists.

According to [1] and [2], $(i_p, W, \sum)$ is an index, called Z_p-index.

Let now W^* be the class of mappings $h : E \to E$ such that:

(1) h is T-equivariant,

(2) h is a homomorphism of the form $e^{tk} + \psi, \psi$ is compact. If $S_\rho = \{u \in E | \ \|u\| = \rho\}$. one defines a pseudoindex i_p^* on the class $\sum$ as:

$$i_p^*(A) = \min\{i_p(h(A) \bigcap E^+ \bigcap S_\rho; \ h \in W^*\}, \quad \forall A \in \sum.$$

According to [2], it is an pseudo-index $(i_p^*, w^*, \sum)$.

Now we shall give the computation of the pseudo-index of invariant space.

Definition 4.3 *An index theory is said to sastiafy the dimension property if there is a positive integer d such that $i_p(V^{dk} \cap S_1) = k$ for all dk-dimensional subspace $V^{dk} \in \sum$ such that $V^{dk} \cap F = \{0\}, F = \{u \in E | T_g u = u, \ \forall g \in G\}$.*

According to Proposition 2.3, we have

Proposition 4.1 *For the Z_p-index $(i_p, W, \sum)$, it has the dimension property with $d = 2$.*

Proof By the Proposition 2.3 in [W1], when $n = 1, \Omega = B_1 = \{u \in E | \|u\| \leq 1\}, b = a = kN$, the proof is complete and $d = 2$.

Then, as consequence of Theorem 2.5 of [2], we have

Proposition 4.2 $i_p^*(H^-) = kN$.

Proof By Theorem 2.5 of [2], $\dim(H^+ \cap H^-) = \dim H_k = 2kN$. $F \cap H^+ = \{0\}, \mathrm{cod}(H^+ + H^-) = \mathrm{cod}E = 0, d = 2$, so $i_p^*(H^-) = kN$.

At last, according to the Theorem 4.2 in [2], when we take the index I and its pseudoindex I_2^* specially as the Z_p-index i_p and i_p^*, then we get

Theorem 4.1 *Suppose f is a functional which satisfies following conditions $f_1 - f_4$, then if the integer $\bar{k}$*

$$\bar{k} = 1/d[\dim(H^+ \bigcap H^-) - cod(H^+ + H^-)]$$

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is well define and positive, the numbers $c_k = \inf_{i_p^*(A)\geq k}\sup_{u\in A} f(u)$ *are critical values of* f *and* $c_0 \leq c_1 \leq \cdots \leq c_k \leq c_\infty$. *Moreover if* $c = c_k = \cdots = c_{k+r}$, *then* $i_p(K_c) \leq r+1$.

(f_1) $f(u) = \frac{1}{2}\langle u, Ku\rangle + \psi(u)$, when K is a bounded selfadjoint operator and ψ' is compact;

(f_2) f sastisfies P.S. condition in $[c_0, c_\infty]$;

(f_3) f is T_g-invariant;

(f_4) a. $F \cap H^+ = \{0\}, F \subset H^-, K(H^-) = H^-, F$ defined in definition 4.3; b. $f'(u) > c_0, \forall u \in H^+ \cap S_\rho$; c. $f(u) < c_\infty, \forall u \in H^-$.

§5. The Proof of Theorem 1.1

According to Theorem 4.1 in section 4, we will prove Theorem 1.1 in this section. First of all, we have to prove conditon (f_2) for I.

Proposition 5.1 *The functional* I *sastisfies P.S. condition, that is ,for any sequence* (v_j) *in* E *such that* $I(v)$ *is bounded and* $I'(v_j) \to 0$ *as* $j \to \infty$, *then it contains a convergent subsequence.*

Proof By Proposition 3.1, we know that 0 is an isolated critical value of f, so according to Lemma 4.8 in [2], the P.S. condition is sastisfied if (v_j) is bounded. But, similar to the proof of [1], if we replace H_k by H, then it can be shown in the same way that (v_j) is bounded.

Now let us give the proof of Theorem 1.1 as follows:

First by Proposition 5.1, we know that condition (f_2) in Theorem 4.1 is satisfied together with (f_1), (f_3), as one easily checks.

Second, condition (a) follw from the definition above, and $c_\infty o f(c)$ follows from Proposition 2.1, $\bar{k} = kN, c_\infty = h(\beta)$.

Let now $H^+ \cap S_\rho = E^+ \cap S_\rho$, where ρ will be chosen in a suitable way in the following. Then, for $z \in E^+$, by the definition of E^+ and $H_2 - H_4$, we have

$$I(z) \geq \frac{1}{2}||z||^2 - \frac{c_2}{\beta}\int_0^{2\pi p} |z|^\beta dt$$

As $H^{1/2}$ is continuously embeded into L^β, then there exist $c_\beta > 0$ that

$$I(z) \geq \frac{1}{2}||z||^2 - \frac{c_2 c_\beta}{\beta}||z||^\beta$$

then, as $z \in S_\rho$, for sufficiently small $\rho > 0$, one has

$$I(z) \geq \frac{1}{2}\rho^2 - \frac{c_2 c_\beta}{\beta}\rho^\beta = c_0 > 0$$

so condition (b) is satisfied. Therefore by Theorem 4.1, the kN numbers $c_k = \inf_{i_p^*(A)\geq k}\sup_{u\in A} I(u)$ are critical values of I and $c_0 \leq c_1 \leq \cdots \leq c_{kN} \leq c_\infty = h(\beta)$. Moreover, if $c = c_k = c_{k+1} = \cdots = c_{k+r}$, then $i_p^*(K_c) \geq r+1$.

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For each critical value $c_i \leq h(\beta), 1 \leq i \leq kN$, it correspondence to a periodic solution of system (1) and have $2\pi p$ as its minimal period by Proposition 3.2. If $c_1 < c_2 < \cdots < c_{kN}$, then I admitts kN distinct solutions of system (1). On the contary, if for some $l \in \{1, \cdots, kN\}, c = c_l = c_{l+1} = \cdots = c_{l+r}, r \geq 1; 1 \leq l < l + r \leq kN$, then we claim that K_c contain infinitely any distinct critical orbits.

In fact, assume that $K_c = \cup_{m=1}^{s} \Theta(u_m)$, where Θu_m is the z_ρ-orbit of u_m, as in [7], it is proved that there is a suitable map $h : K_c \to R \backslash \{0\}$, so that $i_p(K_c) = 1$, then $i_\rho^*(K_c) \leq 1$. But we know that $i_\rho^*(K_c) \geq r + 1 > 1$, so it is impossible, the proof is now complete.

Remark For the general cases of any period pT, the process of proof and the results are the same, so it can be generalized to the general cases.

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非自治Hamilton系统多重次调和最小周期解

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摘 要

本文利用了z_p-指标和伪指标研究了非自治超二次Hamilton系统$J\dot{z} = H_z(z(t), t), z(0) = z(2\pi p)$次调和多重最小周期解，其中$H(z, t + 2\pi p) = H(z, t), t \in R^N, z \in R^{2N}$. 在假设$H_1$–$H_4$条件下，系统至少存在$kN$个不同的以$2\pi p$为最小周期的解.

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