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On Baernstein's Star-Function *

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Let g be a real-valued integrable function on $[-\pi,\pi]$. The Baernstein's star-function of g as given in [1] is

$$g^*(\theta)=\sup_{|E|=2\theta}\int_E g(x)dx, 0\le\theta\le\pi,$$

where $|E|$ denotes the Lebesgue measure of the set $E\subset[-\pi,\pi]$. The *-function has many applications [1-8] to complex function theory. The following theorem plays an essential role in the proof of the sharp inequality (cf. [1])

$$\int_{-\pi}^{\pi}|f(re^{i\theta})|^p d\theta\le\int_{-\pi}^{\pi}|r/\{(1-re^{i\theta})^2\}|^p d\theta, p>0, 0<r<1,$$

for all functions $f=z+\cdots$ regular and univalent in $|z|<1$.

A function ϕ continuous on $-\infty<x<\infty$ is called convex if $\phi(\frac{1}{2}(x+y))\le\frac{1}{2}[\phi(x)+\phi(y)]$, and is called strictly convex if strict inequality holds unless $x=y$. For real-valued $g(x)$, let $[g(x)]^+=\max\{g(x),0\}$.

Theorem (Baernstein) *For $g,h\in L^1[-\pi,\pi]$, the following statements are equivalent.*

(i) *For every convex and nondecreasing function ϕ on $(-\infty,\infty)$,*

$$\int_{-\pi}^{\pi}\phi(g(x))dx\le\int_{-\pi}^{\pi}\phi(h(x))dx. \tag{1}$$

(ii) *For each $t\in(-\infty,\infty)$,*

$$\int_{-\pi}^{\pi}[g(x)-t]^+dx\le\int_{-\pi}^{\pi}[h(x)-t]^+dx. \tag{2}$$

(iii) $g^*(\theta)\le h^*(\theta), \theta\in[0,\pi]$.

We shall discuss the equivalent conditions for strict inequalities in (i)- (iii), and study some properties of the *-function.

As in [1], denote by $\bar{g}$ the symmetric nonincreasing rearrangement function of g. For $g\in L^1[-\pi,\pi]$, let

$$I_g(t)=\int_{-\pi}^{\pi}[g(x)-t]^+dx. \tag{3}$$

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Clearly $I_g(t)$ is continuous on $-\infty < t < \infty$.

Lemma 1 *For $g \in L^1[-\pi,\pi], 0 \le \theta \le \pi, g^*(\theta) = \int_{-\theta}^{\theta} \bar{g}(x)dx$. For each $\theta \in [0,\pi]$, there exists a set $E \subset [-\pi,\pi]$ with $|E| = 2\theta$ for which $g^*(\theta) = \int_E g(x)dx$.*

Lemma 2 *For $g \in L^1[-\pi,\pi]$ and $0 \le a < b < c \le \pi$,*

$$\frac{1}{b-a}\int_a^b \bar{g}(x)dx \ge \frac{1}{c-b}\int_b^c \bar{g}(x)dx.$$

Lemma 1 is due to Baernstein [1]. Lemma 2 follows from the definition of $\bar{g}$.

Lemma 3 *For $g,h \in L^1[-\pi,\pi]$, the equality $g^*(\theta) = h^*(\theta)$ holds on $[0,\pi]$ if and only if the equality $\bar{g}(x) = \bar{h}(x)$ holds a.e. on $[-\pi,\pi]$.*

Proof The sufficiency follows readily from Lemma 1. If $g^*(\theta) = h^*(\theta)$, differentiating both sides and using Lemma 1, we get $(g^*(\theta))' = 2\bar{g}(\theta)$ a.e. in $[0,\pi]$. So is $h^*(\theta)$. Thus $\bar{g}(x) = \bar{h}(x)$ a.e. on $[-\pi,\pi]$.

Theorem 1 *If $g,h \in L^1[-\pi,\pi]$ and satisfy*

$$\int_{-\pi}^{\pi} g(x)dx = \int_{-\pi}^{\pi} h(x)dx \tag{4}$$

then the following statements are equivalent.

(a) For each function $\phi(x)$ strictly convex on $(-\infty,\infty)$,

$$\int_{-\pi}^{\pi} \phi(g(x))dx < \int_{-\pi}^{\pi} \phi(h(x))dx. \tag{5}$$

(b) There exists an interval $I = [a,b], a < b$, such that for almost every $t \in I$,

$$\int_{-\pi}^{\pi} [g(x)-t]^+ dx < \int_{-\pi}^{\pi} [h(x)-t]^+ dx. \tag{6}$$

For t outside I, inequality (2) holds.

(c) There exists an interval $J = [\varphi,\psi], 0 \le \varphi < \psi \le \pi$, such that for almost every $\theta \in J, g^*(\theta) < h^*(\theta)$. For $\theta \in [0,\pi]\backslash J, g^*(\theta) \le h^*(\theta)$.

Proof (a)$\Longrightarrow$(b). Choose a twice differentiable $\phi(x)$ with $\phi''(x) > 0$ on $(-\infty,\infty)$, and $\phi(x) > \phi(0) = 0$ for $x \ne 0$. Write $\phi = \phi_1 + \phi_2$ where $\phi_1(x) = \phi(x)$ for $x < 0$ and vanishes elsewhere. It is easy to see that

$$\phi_1(x) = \int_{-\infty}^{\infty} [t-x]^+ d\phi_1'(t), \phi_2(x) = \int_{-\infty}^{\infty} [x-t]^+ d\phi_2'(t).$$

Interchanging the order of integration, we obtain

$$\int_{-\pi}^{\pi} \phi_2(h(x))dx - \int_{-\pi}^{\pi} \phi_2(g(x))dx = \int_{-\infty}^{\infty} [I_h(t) - I_g(t)]d\phi_2'(t).$$

Interchanging the order of integration and using (4), we have the same equality with ϕ_2 and ϕ_2' replaced by ϕ_1 and ϕ_1' respectively. Thus

$$\int_{-\pi}^{\pi}\phi(h(x))dx-\int_{-\pi}^{\pi}\phi(g(x))dx=\int_{-\infty}^{\infty}[I_h(t)-I_g(t)]d\phi'(t).$$

The left-hand side is positive by the hypotheses (a). Hence (b) follows since $d\phi'(t)>0$ on $(-\infty,\infty)$ and $I_h(t)-I_g(t)\geq 0$ on $(-\infty,\infty)$ by the above Baernstein's Theorem (use a strict nondecreasing ϕ).

(b)$\Longrightarrow$ (a). Let $c\leq a$ at which ϕ is differentiable and set $\mu(t)=\phi'(t-)$. Representating $\phi(x)$ in the form

$$\phi(x)=\phi(c)+(x-c)\phi'(c-)+\int_c^{\infty}[x-t]^+d\mu(t),$$

we have

$$\begin{aligned}\phi(h(x))-\phi(g(x)) &= \phi'(c)(h(x)-g(x))+\int_c^{\infty}[h(x)-t]^+d\mu(t)\\ &\quad -\int_c^{\infty}[g(x)-t]^+d\mu(t).\end{aligned}$$

Using (4) and interchanging the order of integration, we obtain

$$\begin{aligned}\int_{-\pi}^{\pi}\phi(h(x))dx-\int_{-\pi}^{\pi}\phi(g(x))dx &= \int_c^{\infty}[I_h(t)-I_g(t)]d\mu(t)\\ &> \int_a^{b}[I_h(t)-I_g(t)]d\mu(t)\\ &\geq m[\phi'(b')-\phi'(a')]>0,\end{aligned}$$

where a' and b', $a\leq a'<b'\leq b$, are differentiable points of ϕ, and m is the minimal value of $I_h(t)-I_g(t)$ on the interval $[a',b']$.

(b)$\Longrightarrow$(c).Clearly b $\leq$sup ess $\bar{h}(\theta)$, by the properties of a symmetric nonincreasing rearrangement function. Set

$$\varphi=\inf_{0\leq\theta\leq\pi}\{\bar{h}(\theta+)\leq b\leq\bar{h}(\theta-)\},\psi=\sup_{0\leq\theta\leq\pi}\{\bar{h}(\theta+)\leq a\leq\bar{h}(\theta-)\}.$$

If $\varphi=\psi$, the $\varphi>0$ and $I_g(t)<I_h(t)$ a.e. on I. Therefore, there exists a set $F\subset[0,\varphi]$ with $|F|>0$ on which $\bar{g}(x)<\bar{h}(x)$. Let σ be such that $0<|\{[0,\sigma]\cap F\}|<|F|$, and set $J=[\sigma,\varphi]$. Then, for $\theta\in J, g^*(\theta)<h^*(\theta)$. If $\varphi\neq\psi$, we choose $J=[\varphi,\psi]$. By Lemma 1, for $\theta\in J$, there is a set $G\subset[-\pi,\pi]$ of measure 2θ such that

$$g^*(\theta)=\int_G g(x)dx=\int_G[g(x)-t]dx+2\theta t.$$

For $t\in[h(\theta+),h(\theta-)]$,

$$\begin{aligned}\int_G[g(x)-t]dx &\leq \int_{-\pi}^{\pi}[g(x)-t]^+dx<\int_{-\pi}^{\pi}[h(x)-t]^+dx\\ &= \int_{-\pi}^{\pi}[\bar{h}(x)-t]^+dx=\int_{-\theta}^{\theta}[\bar{h}(x)-t]dx.\end{aligned}$$

Thus

$$g^*(\theta) < \int_{-\theta}^{\theta} [\bar{h}(x) - t]dx + 2\theta t = h^*(\theta) \text{ a.e. on } J.$$

(c)$\Longrightarrow$(b). If $\bar{g}(\varphi) \neq \bar{g}(\psi+)$, set $a = \bar{g}(\psi+), b = \bar{g}(\varphi)$ and set $I = [a, b]$. If $\bar{g}(\varphi) = \bar{g}(\psi+)$ and $\bar{g}(\varphi) \geq \bar{h}(\varphi)$, then $\varphi > 0$ and for $\theta \in J, g^*(\theta) < h^*(\theta)$. In this case, there exists a point σ at which $g(x)$ is continuous and $\bar{g}(\sigma) < \bar{h}(\sigma)$. Hence $g^*(\theta) < h^*(\theta)$ for $\sigma \leq \theta \leq \varphi$. Set $a = \bar{g}(\varphi), b = \bar{h}(\sigma)$ and set $I = [a, b]$. In both cases, for $t \in I$ and for θ satisfying $\bar{g}(\theta+) \leq t \leq \bar{g}(\theta)$, the inequalities

$$\begin{aligned}
\int_{-\pi}^{\pi} [g(x) - t]^+ dx &= \int_{-\pi}^{\pi} [\bar{g}(x) - t]^+ dx = \int_{-\theta}^{\theta} [\bar{g}(x) - t]dx = g^*(\theta) - 2\theta t \\
&< h^*(\theta) - 2\theta t = \int_{-\theta}^{\theta} [\bar{h}(x) - t]dx \leq \int_{-\pi}^{\pi} [\bar{h}(x) - t]^+ dx \\
&= \int_{-\pi}^{\pi} [h(x) - t]^+ dx
\end{aligned}$$

hold a.e. on I.

If $\bar{g}(\varphi) = \bar{g}(\psi+)$ and $\bar{g}(\varphi) < \bar{h}(\varphi)$, set $a = \bar{g}(\varphi), b = \bar{h}(\varphi)$ and set $I = [a, b]$. Then for $t \in I$ and θ satisfying $\bar{g}(\theta+) \leq t \leq \bar{g}(\theta)$,

$$\begin{aligned}
\int_{-\pi}^{\pi} [g(x) - t]^+ dx &= g^*(\theta) - 2\theta t \leq h^*(\theta) - 2\theta t = \int_{-\theta}^{\theta} [\bar{h}(x) - t]dx \\
&= \int_{-\theta}^{\theta} [h(x) - t]^+ dx \leq \int_{-\pi}^{\pi} [\bar{h}(x) - t]^+ dx = \int_{-\pi}^{\pi} [h(x) - t]^+ dx.
\end{aligned}$$

The proof is complete.

Theorem 2 *If $g \in L^1[-\pi, \pi]$, then $g^*(\theta)$ is concave on $0 \leq \theta \leq \pi$, and hence it is twice differentiable a.e. in $(0, \pi)$ with $(g^*)''(\theta) \leq 0$.*

Proof For $0 \leq \varphi < \psi \leq \pi$, let $a = \varphi, b = \frac{1}{2}(\varphi + \psi), c = \psi$ and applying Lemma 2, we find

$$\begin{aligned}
g^*(\psi) - g^*(\frac{\varphi + \psi}{2}) &= \int_{-\psi}^{\psi} \bar{g}(x)dx - \int_{-b}^{b} \bar{g}(x)dx \\
&= 2\int_{b}^{\psi} \bar{g}(x)dx - \int_{\varphi}^{b} \bar{g}(x)dx \\
&= \int_{-b}^{b} \bar{g}(x)dx - \int_{-\varphi}^{\varphi} \bar{g}(x)dx \\
&= g^*(\frac{\varphi + \psi}{2}) - g^*(\varphi)
\end{aligned}$$

which completes the proof.

Theorem 3 *Let $g, h \in L^1[-\pi, \pi]$. Then for $\alpha \geq 0, \beta \geq 0$, the equality*

$$(\alpha g + \beta h)^*(\theta) = \alpha g^*(\theta) + \beta h^*(\theta) \tag{7}$$

holds on $0 \leq \theta \leq \pi$ *if and only if the equality*

$$(\alpha g(x) + \beta h(x))^{-} = \alpha \bar{g}(x) + \beta \bar{h}(x) \tag{8}$$

holds a.e. on $[-\pi, \pi]$.

Proof (7)$\Longrightarrow$(8). If $\alpha = 0$ or $\beta = 0$, it is trivial. Now let $\alpha > 0, \beta > 0$. By Lemma 1, for $\theta \in (0, \pi)$,

$$\begin{aligned} \int_0^\theta (\alpha g(x) + \beta h(x))^{-} dx &= \frac{1}{2} \int_{-\theta}^{\theta} (\alpha g(x) + \beta h(x))^{-} dx \\ &= \frac{1}{2} (\alpha g + \beta h) * (\theta) = \frac{1}{2} [\alpha g^*(\theta) + \beta h^*(\theta)] \\ &= \int_0^\theta [\alpha \bar{g}(x) + \beta \bar{h}(x)] dx. \end{aligned}$$

The leftest and the rightest sides are differentiable in $0 < \theta < \pi$, except for a countable set. Hence (8) holds a.e. on $[0, \pi]$. It also holds on $[-\pi, 0]$ by the symmetrility of symmetric nonincreasing rearrangement function.

(8)$\Longrightarrow$ (7). For $0 \leq \theta \leq \pi$, by Lemma 1,

$$\begin{aligned} (\alpha g + \beta h)^*(\theta) &= \int_{-\theta}^{\theta} (\alpha g(x) + \beta h(x))^{-} dx \\ &= \int_{-\theta}^{\theta} [\alpha g(x) + \beta h(x)] dx = \alpha g^*(\theta) + \beta h^*(\theta) \end{aligned}$$

which proves (7) and hence the Theorem.

Remark If g and h are continuous on $[-\pi, \pi]$, then the words "a.e. "in Theorems 1 and 3 can be omitted.

Theorem 4 *If* u *and* v *are subharmonic functions in* $|z| < 1$ *and* u *is subordinate to* v, *then the equality*

$$u^*(re^{i\theta}) = v^*(re^{i\theta}), \theta \in (0, \pi), \tag{9}$$

holds for some $r \in (0, 1)$ *if and only if* $\bar{u}(re^{i\theta}) = \bar{v}(re^{i\theta})$ *holds on* $[-\pi, \pi]$ *and either* v *is harmonic or* $u(z) = v(\lambda z), |\lambda| = 1$.

Proof The sufficiency is easy. To prove the necessity, we apply Lemma 3 to conclude that the equality

$$\bar{u}(re^{i\theta}) = \bar{v}(re^{i\theta}) \tag{10}$$

holds a.e. on $[-\pi, \pi]$.

Since u is subordinate to $v, u(z) = v(w(z))$ where $w(z)$ is analytic in $|z| < 1$ and satisfies $|w(z)| \leq |z|$. Suppose that $w(z) \neq \lambda z, |\lambda| = 1$. Then $|w(z)| < |z|$. If v is not harmonic, let $G(z)$ be harmonic in $|z| < r, r < 1$, and satisfies $G(re^{i\theta}) = v(re^{i\theta})$. By the maximum principle of subharmonic functions, $v(w(z)) < G(w(z))$ on $|z| \leq r$. Thus

$$\int_{-\pi}^{\pi} \bar{u}(re^{i\theta}) d\theta. = \int_{-\pi}^{\pi} u(re^{i\theta}) d\theta = \int_{-\pi}^{\pi} v(w(re^{i\theta})) d\theta$$

$$< \int_{-\pi}^{\pi} G(w(re^{i\theta}))d\theta = 2\pi G(0) = \int_{-\pi}^{\pi} G(re^{i\theta})d\theta$$
$$= \int_{-\pi}^{\pi} v(re^{i\theta})d\theta = \int_{-\pi}^{\pi} \bar{v}(re^{i\theta})d\theta.$$

This contradicts (10). Hence either $u(z) = v(\lambda z), |\lambda| = 1$, or v is harmonic in $|z| < r$. The proof is complete.

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关于 Baernstein 星函数

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摘要

若函数$g(\theta)$在区间$[-\pi,\pi]$上是可积的. 称g在测度为2θ的各子集上的积分的最大值$g^*(\theta)$为g的星函数. 本文考察了星函数的某些性质.