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Some Principal Properties of Generalized Kantorovich Polynomials*

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Let $\alpha_n \geq 0, k \in N$(N be the set of all natural numbers) for any $f \in L_P[0,1]$,

$$M_n^{(k)}(\alpha_n, f, x) \stackrel{\text{def}}{=} (n+k+\alpha_n)^k \sum_{i=0}^{n} \int_0^{\frac{1}{n+k+\alpha_n}} \cdots \int f(\frac{i}{n+k+\alpha_n} + y_1 + \cdots + y_k) dy_1 dy_2 \cdots dy_k P_{n,i}(x) \tag{1}$$

where $P_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}$ $i = 0,1,2,\ldots,n$. $M_n^{(k)}(\alpha_n, f, x)$ is called the generalized Bernstein-Kantorovich polynomials of f. Obviously, when $k = 1, \alpha_n = 0, M_n^{(k)}(\alpha_n, f, x)$ is Bernstein-Kantorovich polynomials of f. We have investigated the rate of convergence of

$$\|M_n^{(k)}(\alpha_n, f, x) - f(x)\|_{L_P[a,b]} \quad 0 < a < a_1 < b_1 < b < 1$$

in terms of the modulus of smooth

$$W_{2,P}(f, h, [a,b]) = \sup_{0<r\leq h} \|f(x+r) + f(x-r) - 2f(x)\|_{L_P[a+r,b-r]} \tag{2}$$

Theorem 1 *For $M_n^{(k)}(\alpha_n, f, x)$ defined by (1), we have*

$$M_n^{(k)}(\alpha_n, 1, x) = 1$$
$$M_n^{(k)}(\alpha_n, t-x, x) \asymp \frac{1+\alpha_n}{n}$$
$$M_n^{(k)}(\alpha_n, (t-x)^2, x) \asymp \frac{1}{n} + (\frac{\alpha_n}{n})^2$$

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Proof The proof is trivial.

Theorem 2 *Let $f \in L_P[0,1]$, $1 \le P \le \infty$, $[a_1, b_1] \subset [a,b]$. Then for any $l > 0$*

$$\|M_n^{(k)}(\alpha_n, \chi_{C[a,b]} \cdot f)\|_{L_P[a_1,b_1]} = O\{\left(\frac{1+\alpha_n}{n}\right)^l \|f\|_P\} \tag{3}$$

where

$$\chi_{[a,b]} = \begin{cases} 1 & x \in [a,b] \\ 0 & x \notin [a,b] \end{cases}.$$

Proof For $u \notin [a,b], x \in [a_1, b_1]$ we have $|u - x| \ge \delta > 0$. For $P > 1$, by Hölder inequality, since $M_n^{(k)}(\alpha_n, \cdot)$ is uniform bounded positive linear operator, then

$$\begin{aligned} &\|M_n^{(k)}(\alpha_n, \chi_{C[a,b]} f)\|_{L_P[a_1,b_1]} \\ &\le \|(M_n^{(k)}(\alpha_n, |\chi_{C[a,b]}|^q, x))^{\frac{1}{q}}\|_\infty \cdot \|(M_n^{(k)}(\alpha_n, |f|^P, x))^{\frac{1}{P}}\|_P \\ &\le C_1(k,m)\{\left(\frac{1+\alpha_n}{n}\right)^m + n^{-\frac{m}{2}}\}\|f\|_P. \end{aligned}$$

Choosing $m = 2l$, we complete the proof for $P > 1$. For $P = 1$

$$\begin{aligned} &\|M_n^{(k)}(\alpha_n, \chi_{C[a,b]} f)\|_{L_P[a_1,b_1]} \\ &\le (n+k+\alpha_n)^{n-1} \int_0^{\frac{1}{n+k+\alpha_n}} \cdots \int \{\max_{a_1 \le x \le b_1} P(n,k,\alpha_n,y_1,\cdots,y_k,x) \\ &\chi_{C[a,b]}(y_1)\} dy_2 \cdots dy_k \|f\|_L. \quad y_1 \in [y_2 + \cdots + y_k, \frac{n+1}{n+k+\alpha_n} + y_2 + \cdots + y_k] \end{aligned}$$

for $y_1 \notin [a,b], x \in [a,b]$, we have $|y_1 - x| \ge \delta > 0$

$$\begin{aligned} &\max_{a_1 \le x \le b_1} P(n,k,\alpha_n,\cdots,y_k,x)\chi_{C[a,b]}(y_1) \\ &\le \frac{1}{\delta^m}(n+k+\alpha_n)c(m)\frac{1}{n^m} \max_{a_1 \le x \le b_1} P_{n,i}(x) \\ &\le c(k,m)n^{-m+1}, \quad y_1 \in [y_2 + \cdots + y_k + \frac{i}{n+k+\alpha_n}, y_2 + \cdots + y_k + \frac{i+1}{n+k+\alpha_n}], \end{aligned}$$

we have

$$\begin{aligned} &\|M_n^{(k)}(\alpha_n, \chi_{C[a,b]} f)\|_{L[a_1,b_1]} \\ &\le (n+k+\alpha_n)^{k-1} \int_0^{\frac{1}{n+k+\alpha_n}} \cdots \int \{\max_{0 \le i \le n} \max_{a_1 \le x \le b_1} P_{n,i}(x)\frac{1}{\delta^m}|y_1 - x|^m (n+k+\alpha_n)\} \\ &dy_2 \cdots dy_k \|f\|_L \le c(k,m)n^{-m+1}\|f\|_L, \end{aligned}$$

where $y_1 \in [y_2 + \cdots + y_k + \frac{i}{n+k+\alpha_n}, y_2 + \cdots + y_k + \frac{i+1}{n+k+\alpha_n}]$. Choose $m = l + 1$ we obtain the proof for $P = 1$.

Theorem 3 *For $\beta \geq \gamma > 0$, there exist constant $c \equiv c(k,\beta,\gamma)$, such that for any l and t* $(l > 0)$

$$f_t(u) \overset{\text{def}}{=} \begin{cases} |u-t|^\gamma & |t-n| \geq ln^{-\frac{1}{2}} \\ 0 & |t-n| < ln^{-\frac{1}{2}} \end{cases}$$

$$M_n^{(k)}(\alpha_n, f_t(u), t) \leq c(k,\beta,\gamma) l^{\gamma-\beta} n^{-\frac{\gamma}{2}} \{1 + n^{-\frac{\beta}{2}}(1+\alpha_n)^\beta\} \tag{4}$$

Proof Since $|u-t| \geq ln^{-\frac{1}{2}}$, we use Lemma 3 of [2] and obtain

$$\begin{aligned} M_n^{(k)}(\alpha_n, f_t(u), t) &\leq l^{-\beta+\gamma} n^{\frac{\beta-\gamma}{2}} M_n^{(k)}(\alpha_n, |u-t|^{\beta-\gamma} f_t(u), t) \\ &\leq l^{-\beta+\gamma} n^{\frac{\beta-\gamma}{2}} c(k,\beta)\{(\frac{1+\alpha_n}{n})^\beta + n^{-\frac{\beta}{2}}\} \\ &\leq c(k,\beta) l^{\gamma-\beta} n^{-\frac{\gamma}{2}} \{1 + n^{-\beta/2}(1+\alpha_n)\}. \end{aligned}$$

Theorem 4 *For any $f \in L_P[0,1], 1 \leq P \leq \infty$, there exists a function $g_h, g_h'' \in L_P[a,b]$, such that*

$$\|f - g_h\|_{L_P[a,b]} \leq \frac{1}{2} w_{2,P}(f,h,[a,b]), \tag{5}$$

and

$$\begin{aligned} \|g_h'\|_{L_P[a,b]} &\leq \frac{2}{h} w_{1,P}(f,h,[a,b]), \\ \|g_h''\|_{L_P[a,b]} &\leq h^{-2} w_{2,P}(f,h,[a,b]). \end{aligned} \tag{6}$$

Proof By [1], in fact only if let

$$g_h(x) = \frac{1}{2h^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} |f(x+u+v) - f(x-u-v)| du dv.$$

Theorem 5 *Let $f \in L^2[a,b]$. Then*

$$\begin{aligned} &\frac{n}{1+\alpha_n}[M_n^{(k)}(\alpha_n, f, x) - f(x)] \\ &= \frac{1}{1+\alpha}\{[\frac{k}{2}(1-2x) - \alpha x] f'(x) + \frac{1}{2}x(1-x)f''(x)\} + o(1), \end{aligned} \tag{7}$$

uniformly for x.

Theorem 6 *Let $f \in L_P[0,1], g \in c^2$, supp $g \subset [a,b]$. Then we have*

$$\frac{n}{1+\alpha_n} | < M_n^{(k)}(\alpha_n, f, x) - f(x), g(x) > | \leq c(\|g\|_c + \|g'\|_c + \|g''\|_c)\|f\|_{L_P[0,1]}. \tag{8}$$

Proof Using Taylor's formula, we have

$$\begin{aligned} &\frac{n}{1+\alpha_n} < M_n^{(k)}(\alpha_n, f, t) - f(t), g(t) > \\ &= \frac{n}{1+\alpha_n}\{-\int_0^1 f(t)g(t)dt + \int_0^1 \sum_{i=0}^n (n+k+\alpha_n) \int_0^{\frac{1}{n+k+\alpha_n}} \cdots \int f(\frac{i}{n+k+\alpha_n} \end{aligned}$$

$$+ y_1 + \cdots + y_k)g(\frac{i}{n+k+\alpha_n} + y_1 + \cdots + y_k)P_{n,i}(t)dy_1 \cdots dy_k \cdot dt\}$$

$$+ \frac{n}{1+\alpha_n}\{\int_0^1 \sum_{i=0}^{n}(n+k+\alpha_n)^k \int_0^{\frac{1}{n+k+\alpha_n}} \cdots \int f(\frac{i}{n+k+\alpha_n} + y_1 + \cdots + y_k)$$

$$\cdot g'(\frac{i}{n+k+\alpha_n} + y_1 + \cdots + y_k)(t - \frac{i}{n+k+\alpha_n} - y_1 - \cdots - y_k)P_{n,i}(t)dy_1 \cdots dy_k dt\}$$

$$+ \frac{n}{2(1+\alpha_n)}\{\int_0^1 \sum_{i=0}^{n}(n+k+\alpha_n)^k \int_0^{\frac{1}{n+k+\alpha_n}} \cdots f(\frac{i}{n+k+\alpha_n} + y_1 + \cdots + y_k)$$

$$\cdot g''(\xi)(t - \frac{i}{n+k+\alpha_n} - y_1 - \cdots - y_k)^2 P_{n,i}(t)dy_1 \cdots dy_k dt\}$$

$$= I_1 + I_2 + I_3.$$

For n sufficiently large, then $I_1 = c\|g\|_c\|f\|_{L^P[0,1]}, |I_2| \leq c(k)\|f\|_{L[0,1]}\|g'\|_c$.

In the same way we obtain $I_3 \leq c(k)\|g''\|_c\|f\|_{L[0,1]}$, complete the proof of Theorem 6.

References

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推广的Kantorovich 多项式的一些基本性质

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摘 要

本文对推广的Bernstein-Kantorovich 多项式进行了深入地研究和讨论，给出并证明了一些重要的基本性质.