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A Proof of 3-dimensional Poincaré Conjecture*

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Abstract A Heegaard splitting of an orientable closed cnnected 3-manifold M is a closed connected surface $F \hookrightarrow M$ such that M is divided into two handlebodies. Let $g(M)$ be the minimal genus of all such surfaces. Let $r(M)$ be the rank of $\pi_1(M)$. Then $r(M) \leq g(M)$. Waldhausen ([3] p.320) asked whether $r(M) = g(M)$ is true for all M. But Boileau and Zieschang gave a negative answer to the question by describing some Seifert manifold M with $2 = r(M) < g(M) = 3$ ([4]). In this paper, however, we shall prove that if $\pi_1(M)$ is trivial, then $g(M) = r(M)$, thus M has a Heegaard splitting with genus 0, i.e. M is a 3-sphere. This is the assertion which Poincare' conjectured in 1904. There are to approaches to the Poincare' conjecture, but here we shall work on it through its Heegaard splitting.

§1 Rearrangement of words theorem

A word is a finite sequence of letters, $w = b_1 \cdots b_m, m \geq 0, b_i \in X^{\pm 1}, X = \{x_1, \cdots, x_n\}$, $1 \leq i \leq m$. If $m = 0$, then $w = 1$, the empty word. The set $W = W(X)$ of all words is a semigroup under juxtaposition. A word w is reduced if it contains no part bb^{-1} and cyclical reduced if it is reduced and $b_1 \neq b_m^{-1}$. For $u = u_1u_2 \in W$, we say that u_2u_1 is a rotation of u and write $\sigma(u) = u_2u_1$; for $a, b \in W$, we denote "b inserting in a" by $a \wedge b$, i.e. $a = a_1a_2$ and $a \wedge b = a_1ba_2$. Obviously the products ab and ba are special cases of b insertion in a.

1.1 Definition *For $a_1, \cdots, a_k \in W(X)$, an element $\psi_k(a_1, \cdots, a_k) \in W$ is defined as follows:*

1) $\psi_1(a_1) = a_1, \psi_2(a_1, a_2) = a_1 \wedge \sigma(a_2)$ *for some rotation $\sigma(a_2)$ of a_2;*

2) $\psi_k(a_1, \cdots, a_k) = \psi_{k-1}(b_1, \cdots, b_{k-1})$, *where either $b_1 = a_1 \wedge \sigma(a_{i_k}), b_j = a_{i_j}, 2 \leq j \leq k-1$, or $b_1 = a_1, b_j = a_{i_j}, j \neq h, k, 1$, and $b_h = a_{i_h} \wedge \sigma(a_{i_k}), h \neq k, h, k \geq 2, \{i_2, \cdots, i_k\} = \{2, \cdots, k\}$.*

1.2 Theorem *For any $a_1, \cdots, a_k \in W(X)$, if*

$$x_1 = y_1 a_1 y_1^{-1} \cdots y_k a_k y_k^{-1} \tag{1}$$

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in $F(X)$ with $y_i \in F(X)$, then there is an equation

$$x_1 = \psi_k(a'_1, \cdots, a'_k) \tag{2}$$

in $F(X)$ where $a'_i = \sigma_i(a_{j_i}), 1 \le i \le k, \{j_1, \cdots, j_k\} = \{1, \cdots, k\}$.

1.3 Theorem (Free basic points theorem) *For any $a_1, \cdots, a_k \in W(X)$ and $c_i \in W(X), 1 \le i \le k$, if*

$$x_1 = y'_1 c_1 a_1 c_1^{-1} {y'_1}^{-1} \cdots y'_k c_k a_k c_k^{-1} {y'_k}^{-1} \tag{3}$$

in $F(X)$, then there is another equation (2) as in 1.2.

§2 The band connected sums of simple closed curves on ∂V

Let V be an oriented handlebody with genus $n, \{B_1, \cdots, B_n\}$ a collection of pairwise disjoint, properly embedded 2-cells in V such that $V - \cup_{i=1}^n B_i \times (-1, 1)$ is a 3-cell. If each B_i is oriented, then such a collection is called a basis of V and written as $B = \{B_1, \cdots, B_n\}$.

Let $J \in \partial V$ be an oriented simple closed curve with a basic point $x_0 \in J - B$. We consider the intersecting points of J with $B_i, 1 \le i \le n$, from x_0 to x_0 along the direction of J. If they intersect positively, we denote the intersecting point by x_i; otherwise by x_i^{-1}. Thus we obtain their juxtaposition $\varphi_B(J)$. Similarly we can define $\varphi_B(I)$ for an oriented arc I on ∂V. Let $y \in \partial V$ be a basic point of $\pi_1(V), \varphi_B([y, x_0]) = c$. Then $c\varphi_B(J)c^{-1} = [J] \in \pi_1(V) = F(X)$. For a collection $H = \{J_1, \cdots, J_k\}$ of pairwise disjoint simple closed curves on $\partial V, \varphi_B(J_i) = a_i, c_i a_i c_i^{-1} = [J_i] = b_i \in \pi_1(M), 1 \le i \le k$. If there is an equation

$$x_1 = y_1 b_1 y_1^{-1} \cdots y_k b_k y_k^{-1}$$

in $F(X)$, then by 1.3, we have

$$x_1 = \psi_k(a'_1, \cdots, a'_k) = \psi_k(a_1, \cdots, a_k) \tag{4}$$

in $F(X)$. This is the foundation of all our discussions in the paper.

Let $H = \{J_1, \cdots, J_k\}$ be the same as above, $p_h \in J_h, h = i, j (i \ne j)$. We define the band connected sum $J_i \# J_j$ between J_i and J_j along a simple arc $[p_i, p_j]$ calculus in H as

$$J_i \# J_j = (J_i - p_i \times (-1,1)) \cup (p_i, p_j) \times (-1) \cup (J_j - p_j \times (-1,1)) \cup (p_i, p_j) \times 1$$

where "calculus in H" means $(p_i, p_j) \subset \partial V - H$.

2.1 Definition *A band connected sum $J_i \# J_j$ is called regular if $\varphi_B([p_i, p_j]) = 1$.*

2.2 Proposition *Suppose that*

i) the right side of (4) contains a part $\psi = u x_h^\epsilon v x_h^{-\epsilon} \epsilon w$, i.e. $\psi_k = Z_1 \psi Z_2$;

ii) $x_h^\epsilon \epsilon J_i, x_h^{-\epsilon} \in J_j$ is a pair of cancellation points in (4);

iii) φ contains a_i or a_j and there is a regular band connected sum $J_i \# J_j$ in H.

Then there is a $\psi' \in W(X)$ such that

1) ψ' and ψ have the same initial and end letters;

2) $\psi' = \psi$ in $F(X)$;
3) if we replace ψ by ψ' in ψ_k, then in $F(X)$, (4) becomes

$$x_1 = \psi_{k-1}(a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_{j-1}, a_{j+1}, \cdots, a_k, \varphi_B(J_i \# J_j)).$$

§3 Genus reducing curves on ∂V

3.1 Definition *Let V be an orientable handlebody with genus n. A simple closed curve J on ∂V is called to be a genus reducing curve, if*

$$V(J) = V \cup B' \times [-1,1]$$

is a handlebody with genus $n-1$, where B' is a 2-cell, and in $V(J)$ $V \cap B' \times [-1,1] = \partial V \cap \partial B' \times [-1,1] = \partial B' \times [-1,1], \partial B' \times \{0\} = J$.

3.2 Theorem *A simple closed curve J on ∂V is the genus reducing curve if and only if there is a 2-cell $B_i^* \hookrightarrow V$ such that $J \cap B_i^*$=single point.*

In fact, the sufficiency is obvious and we shall use the sufficiency in the paper.

3.3 Proposition *Let $J \subset \partial V$ be a simple closed curve, B a base of V. If $\varphi_B(J) = x_1$ in $F(X)$, then J is a genus reducing curve.*

Let $H = (J_1, \cdots, J_k), \varphi_B(J_i) = a_i, 1 \le i \le k$, as before. Assume they satisfy (4) as in §2. We now observe geometrically cancelling process on the right side of (4). Let Σ be such a process (although this process can be not unique) in which the right side of (4) becomes the left side. Then for each j, the points of $\partial B_j \cup B$ are pairwisely cancelled under Σ, except the x_1 on the left side of (4). We denote these pairs by $p_i, p_i^l, 1 \le i \le h_j$; and denote the arc in ∂B_j with two end points p_i, p_i^l by $[p_i, p_i^l]$.

3.4 Definition i) *A pair p_i, p_i^l of points is said to be separating, if there is another pair $p_r, p_r^l (1 \le r \le h_j)$ such that only one of them is in the interior of $[p_i, p_i^l]$; Otherwise, p_i, p_i^l is said to be nonseparating;*

ii) *Σ is said to be nonseparating if each pair p_i, p_i^l is nonseparating, for $1 \le i \le h_j, 1 \le j \le n$.*

3.5 Lemma *Let $H_k = \{J_1, \cdots, J_k\}$ as before. Assume that H_k satisfies a nonseparating equation (4), then H_k can be changed into $H_{k-1} = \{J_1^l, \cdots, J_{k-1}^l\}$ such that H_{k-1} also satisfies the equation (4)(in the case $k-1$), where $J_i^l (1 \le i \le k-1)$ is the band connected sum of curves in H_k.*

3.6 Lemma *Let $H_k = \{J_1, \cdots, J_k\}$ as before and it satisfy a separating equation (4) and $\varphi_B(J_i) = a_i, 1 \le i \le k$ are cyclically reducings. Then there is a 2-cell $B_i^* \hookrightarrow V$ which is a band connected sum of members of B such that $B_i^* \cap$ some J_j =single point.*

Remark By 3.5 and 3.6, if H_k satisfies (4), then H_k "contains" a genus reducing curve, which is either a member of H_k or a band connected sum of the members of H_k. As a counterexample the conclusion, we now introduce a note on a result of Boileau-Zieschang. In [8] Montesinos has given a Heegaard diagram $(V; J_1, J_2, J_3)$, where $\varphi_B(J_1) = x_2x_1^{-1}x_2x_1^{-3}, \varphi_B(J_2) = x_3x_1x_3x_1^{-1}$ and $\varphi_B(J_3) = (x_3x_2x_1^{-1})^3(x_2x_1^{-1})^2$. Let $r_i = \varphi_B(J_i), 1 \leq i \leq 3$. By [8], we have

$$x_3x_2x_3^{-1}x_2^{-1}x_3x_2^2x_1^{-1}x_2 = \psi_9(a_1, \cdots, a_9), \quad (4)*$$

where $a_i = \sigma_i(r_{j_i}^{\pm1}), 1 \leq i \leq 9$. Is there a trasformation

$$T : x_1^{*-1} \to x_3x_2x_3^{-1}x_2^{-1}x_3x_2^2x_1^{-1}x_2$$

such that it carries (4)* into (4)

$$x_1^{*-1} = \psi_9(T^{-1}(a_1), \cdots, T^{-1}(a_9))?$$

and keep $T^{-1}(a_i)$ to be cyclical reduced. No there isn't. Because there exists no base B^* of V such that $\sigma_i(\varphi_{B^*}(J_{j_i}^{\pm1})) = T^{-1}(a_i)$ are cyclical reduced. Hence the generalized Nielsen operations (see [8]) in the paper can't make sense.

3.7 Theorem *Let M be a connected, simply connected, closed 3- manifold. If M has a Heegaard splitting with genus n, then it has a Heegaard splitting with genus $n-1$. Hence by the induction M is homeomorphic to S^3.*

3维Poincaré猜想的一个证明

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摘要

设M是一个连通闭3维流形而且$\pi_1(M) = \langle x_1, \cdots, x_n; y_1, \cdots, y_n \rangle = 1$. 在本文中，我们利用由$\pi_1(M) = 1$得出来的条件$x_1 = y_1a_1y_1^{-1}\cdots y_ka_ky_k^{-1}$(其中$y_j \in F(x_1, \cdots, x_n), a_j = r_{ij}^{\pm}, 1 \leq j \leq k$)给出著名的Poincaré猜想一个肯定的回答，即这样的M必然同胚于S^3.