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Axiomatizing Trigonometry *

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Abstract In this paper, two axioms for cosine function are given and the well-known definitions defined by power series as well as by the arc length on a unit circle have been obtained. Obviously the latter implies both axioms. Thus the two axioms given do constitute a characterization for the cosine function.

$F \in R_R$ is a cosine function if it satisfies

Axiom I $\exists \alpha \quad \forall x, y, (F(x-y) = F(x)F(y) + F(\alpha+x)F(\alpha+y));$

Axiom II $\lim\limits_{x\to 0} \dfrac{F(\alpha+x)}{x} = -1.$

Definition I $G(x) = D_f - F(\alpha+x)$ *is called a sine function*

Axiom I' $\forall x, y(F(x-y) = (F(x)F(y) + G(x)G(y));$

Axiom II' $\lim\limits_{x\to 0} \dfrac{G(x)}{x} = 1.$

Theorem 1 $\exists \beta, \quad (F(\beta) \neq 0).$

Theorem 2 $(1)\forall x(F^2(x) + G^2(x) = F(0));\ (2)\gamma = F(0) = F^2(\beta) + G^2(\beta) > 0.$

Theorem 3 $(1)|G(0)| = (\gamma - \gamma^2)^{1/2};\ (2)\gamma \in (0,1].$

Theorem 4 $\forall x(F(-x) = F(x)).$

Proof $\forall y, z(F(y-z) = F(z-y)).$

Theorem 5 $(1)G(\alpha) = 1;(2)\gamma = 1.$

Proof $(1)\forall x(G(x) = -F(\alpha+x) = -F(\alpha)F(-x) - G(\alpha)G(-x)),\quad \forall x(G(-x) = -F(\alpha - x) = -F(\alpha)F(-x) - G(\alpha)G(x)) \Longrightarrow \forall x(G(x) - G(-x) = G(\alpha)(G(x) - G(-x))), \forall x\ (G(x) = G(-x) \Longrightarrow \forall x(F(x) = F(\frac{x}{2})F(-\frac{x}{2}) + G(\frac{x}{2})G(-\frac{x}{2}) = F^2(\frac{x}{2}) + G^2(\frac{x}{2}) = \gamma) \Longrightarrow$
$\lim\limits_{x\to 0} \dfrac{F(x+\alpha)}{x} = \infty \otimes \Longrightarrow G(\alpha) = 1.$

(2) $1 \geq r = F^2(\alpha) + G^2(\alpha) \geq 1.$

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Theorem 2' $\forall x(F^2(x)+G^2(x)=1)$.

Theorem 6 $\forall x(G(-x)=-G(x))$.

Proof $G(-x)=-F(\alpha-x)=-F(\alpha)F(x)-G(\alpha)G(x)=-G(x)$.

Theorem 7 $\forall x,y(F(x+y)=F(x)F(y)-G(x)G(y))$.

Theorem 8 $\forall x(G(\alpha+x)=F(x))$.

Proof $G(\alpha+x)=-G(-\alpha-x)=F(-x)=F(x)$.

Theorem 9 $\forall x,y(G(x\pm y)=G(x)F(y)\pm F(x)G(y))$.

Proof $G(x+y)=-F(\alpha+x+y)=-F(\alpha+x)F(y)+G(\alpha+x)G(y)=G(x)F(y)+F(x)G(y)$.

Theorem 10 (1)$\forall x$ $\left\{\begin{matrix}F\\G\end{matrix}\right.(2n\alpha+x)=(-1)^n\left\{\begin{matrix}F\\G\end{matrix}\right.(x)$;

(2) $\forall x$ $\left\{\begin{matrix}F\\G\end{matrix}\right.(\overline{2n+1}\alpha-x)=(-1)^n\left\{\begin{matrix}G\\F\end{matrix}\right.(x)$.

Proof (1) $F(2(n+1)\alpha+x)=-G(\overline{2n+1}\alpha+x)=-F(2n\alpha+x)$.
Theorem 10 entails that F,G are periodic functions with periodicity 4α.

Theorem 11 $\forall x,y$, (1)$F(x)F(y)=\frac{1}{2}(F(x-y)+F(x+y))$; (2)$G(x)G(y)=\frac{1}{2}(F(x+y)-F(x-y))$; (3)$G(x)F(y)=\frac{1}{2}(G(x+y)+G(x-y))$.

Theorem 12 $\forall x,y$, (1)$F(x)+F(y)=2F(\frac{x+y}{2})F(\frac{x-y}{2})$; (2) $F(x)-F(y)=-2G(\frac{x+y}{2})G(\frac{x-y}{2})$; (3) $G(x)+G(y)=2G(\frac{x+y}{2})F(\frac{x-y}{2})$.

Theorem 13 (1)$\lim\limits_{x\to 0}G(x)=0$; (2) $\lim\limits_{x\to x_0}F(x)=F(x_0)$; (3) $\lim\limits_{x\to x_0}G(x)=G(x_0)$.

Theorem 14 $\forall x$, (1)$F'(x)=-G(x)=F(\alpha+x)$; (2)$G'(x)=F(x)=G(\alpha+x)$; (3)$F^{(n)}(x)=F(n\alpha+x)$; (4)$G^{(n)}(x)=G(n\alpha+x)$.

Theorem 15 $\forall x$, (1)$F(x)=\sum_0^\infty(-1)^n\frac{x^{2n}}{(2n)!}$; (2)$G(x)=\sum_0^\infty(-1)^n\frac{x^{2n+1}}{(2n+1)!}$; (3)$e^{ix}=F(x)+iG(x)$.

Theorem 16 $\exists 1\delta>0$, $(F(\delta)=0\wedge\forall x\in(0,\delta),(F(x)>0))$.

Proof Let $S=F^{-1}(0)\cap[0,|\alpha|]$ which is a non-empty closed bounded set

$$\Longrightarrow \exists 1\delta=\wedge S\in S\Longrightarrow F(\delta)=0\wedge\delta>0,\ \exists\eta\in(0,\delta),(F(\eta)\le 0)$$
$$\Longrightarrow \exists\varepsilon\in(0,\eta]\subseteq(0,\delta),(F(\varepsilon)=0)\quad\otimes.$$

Theorem 17 *4δ is the least positive period of F and G.*

Proof (1)$F(2\delta)=2F^2(\delta)-1=-1, G(2\delta)=0\Longrightarrow F(4\delta)=2F^2(2\delta)-1=1, G(4\delta)=0$ $\Longrightarrow\forall x$, $(F(4\delta+x)=F(4\delta)F(x)-G(4\delta)G(x)=F(x))\Longrightarrow 4\delta$ is a positive period of F.
(2) $\forall x\in x(0,\delta)$ $G'(x)=F(x)>0\Longrightarrow G(x)\nearrow\Longrightarrow G(x)>0$, $F'(x)=-G(x)<$

$0 \Longrightarrow F(x) \searrow \Longrightarrow F(x) \in (0,1), \exists \varepsilon \in (0,4\delta) \forall x \in R(F(\varepsilon + x) = F(x)) \Longrightarrow \frac{\varepsilon}{4} \in (0,\delta) \wedge$ $2(2F^2(\frac{\varepsilon}{4}) - 1)^2 - 1 = F(\varepsilon) = F(0) = 1$

$\Longrightarrow F^2(\frac{\varepsilon}{4})(F^2(\frac{\varepsilon}{4}) - 1) = 0 \Longrightarrow F(\frac{\varepsilon}{4}) = 0$ or 0 or $\pm 1 \otimes$.

Theorem 18 (1)$C : \left\{ \begin{array}{l} x = F(t) \\ y = G(t) \end{array} \right\} \to t =$ the directed arc length joining $T_0 = \langle 1,0 \rangle$ and $T_t = \langle F(t), G(t) \rangle$ on the curve C;

(2) $\delta = \pi/2$.

Proof (1)$l_{T_0T_t} = \int_0^t \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt = \int_0^t \sqrt{G^2(t) + F^2(t)} dt = t.$

(2) $4\delta = l_{T_0T_{4\delta}} = 2\pi$.

It means that the functions F, G coincide with the functions cos and sin respectively, i.e.. The function cos is uniquely determined by the axioms I and II.

三角学的公理化

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摘 要

本文给出余弦函数的两个公理:

公理1 $\exists \alpha \ \forall x, y \quad (F(x-y) = F(x)F(y) + F(\alpha + x)F(\alpha + y))$;

公理2 $\lim_{x \to 0} \frac{F(\alpha + x)}{x} = -1$.

如果取 $\alpha = \pi/2$, 则公理1,2是三角学中两条定理. 本文通过证明

定理15 $\forall x$ (1) $F(x) = \sum_0^\infty (-1)^n \frac{x^{2n}}{(2n)!}$; (2) $F(\alpha + x) = \sum_0^\infty (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!}$ (文中为简略起见把 $-F(\alpha + x)$ 写作 $G(x)$,(2) 说明 $G(x)$ 仅与 x 有关.)

定理16 $\exists 1 \quad \delta > 0 \quad (F(\delta) = 0 \wedge \forall x \in (0,\delta)(F(x) > 0))$.

定理18 (1) $C \left\{ \begin{array}{l} x = F(t) \\ y = G(t) \end{array} \right\} \to t = C$ 上联结 $T_0 = \langle 1,0 \rangle$ 与 $T_t = \langle F(t), G(t) \rangle$ 的有向弦长;

(2) $\delta = \frac{\pi}{2}$.

得到公理1,2等价于我们熟知的余弦函数定义域幂级数定义. (见Dr.Konrad Knopp) heory and application of Inpinitc sehies (Frist English Edition 1928; Second English Edition 1951)).