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The Local Asymptotic Behavior of Solutions of Second Order Linear Differential Equations in a Nutshell*

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1. Introduction

Given a second order linear differential equation with meromorphic coefficient

$$a(x)y'' + b(x)y' + c(x)y = 0 \tag{1.1}$$

in a domain D the local analytic theory for such equations is well developed, see e.g. [8].

If x_0 is a regular point for (1.1) then the theory advocates to utilize Taylor series expansions of two linearly indepentdent solutions of (1.1). If x_0 is a regular singular point for (1.1), Frobenius' method tells us to follow sometimes a laborious procedure which produces two linearly independent solutions of (1.1). If x_0 is a singular irregular point for (1.1), some elementary books still skip this case, e.g. [2]. Other books suggest to utilize the celebrated Liouville-Green approximation (or the WKB method). In other words we are advised to classify a point x_0 and according to this classification we are asked to apply one of three essentially different methods. If x_0 is a turning point, we need more sophisticated methods.

The purpose of this article is to propose a different approach which is based on physical motivation. Essentially we propose to utilize two theorems, developed in [3] and [4] which have the advantage that at each point x_0 of interest, two linearly independent solutions of (1.1) are given which represent an incident and a reflected wave, regardless if x_0 is a regular, a singular regular, a singular irregular ro even a turning point for (1.1). The formulas of course are valid even in cases where $a(x), b(x)$ and $c(x)$ are not necessarily meormorphic. However, in order to aviod verbiage, we will assume that $a(x), b(x)$ and $c(x)$ are meromorphic.

In the sequel we will transform equation (1.1) into a canonical form compatible with a "Schrodlinger type" equation of the form

$$z'' = \phi(x)z, \tag{1.2}$$

where z is the dependent variable.

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The theorems that we are going to use are designed for the equation (1.2).

To demonstrate our method we choose a special equation which possesses a variety of different types of singularities. We will then show how at most points of interest for that special equation just one set of linearly independent solutions can be given.

In order to convince the reader about the usefulness of our approach, let us first describe two benefits whcih do not involve many technicalities. One benefit in oscillation theory and the other in acoustic wave propagation.

Thanks to the fact that we were able to reduce the asymptotic behaviors of solutions of second order equations to the consideration of just two theorems, a simple necessary and sufficient condition for oscillation of solutions of (1.2) is possible. Namely, let $\phi(x)$ be a real meromorphic function at $x = b$, (b finite or not), then a nontrivial solution of (1.2) possesses infinitely manyzeros with accumulation point at b if and only if the function

$$B(x) := \mathrm{Im}\pi^{-1} \int^{x} \sqrt{\phi + (\frac{1}{4}\frac{\phi'}{\phi})^2}\, dt \tag{1.3}$$

is unbounded as $x \to b^-$ (or $x \to b^+$). This is proved in [3,4,5].

The two theorems mentioned here lead to a new natural normalization of incident and reflected waves. A local analysis of these wave functionas lead to a few local principles of wave propagation in inhomogeneous media. It turns out that

a) Reflection increases as inhomogeneity increases.

b) Amplitude of oacillation increases as inhomogeneity increases.

c) The amount of oscillation of a wave function decreases as inhomogeneity increases.

d) The transmission is a monotone increasing function of the time frequency. This is due to [1].

2. The main theorems

References [3] and [4] give two sets of invariant asymptotic formulas of the second order differential equation

$$y'' = \phi(x)y \tag{2.1}$$

with $\phi(x)$ being a meromorphic function of x on $[a, b]$.

A function called the "comparability index" $l(x)$, is defined as follows.

$$l(x) = \frac{\phi'(x)}{4\phi^{3/2}(x)}. \tag{2.2}$$

This is a mapping which provides important information on the asymptotic nature of solutions of (1.2). For example, if $l^2(x) \neq -1$ on $[a, b]$, then the general solution is determined by [3], if $l^2(x) = -1$ at $x = b$, then the general solution is given by [4]. This implies that in "most cases", namely $l^2(x) \neq -1$, the local asymptotic theory is covered by just one set of formulas.

Let us define the following functions which will appear in the solutions given in Theorem 2.1 and Theorem 2.2.

$$\lambda(x) = \sqrt{\phi + (\frac{1}{4}\frac{\phi'}{\phi})^2} \tag{2.3}$$

and

$$r(x) = -\frac{i}{2}\frac{l'(x)}{(1+l^2(x))} = \frac{i}{2}\frac{(l^{-1}(x))'}{(1+l^{-2}(x))}. \tag{2.4}$$

From [3], one set of solution is given as

Theorem 2.1 *Let $\phi(x)$ be meromorphic in interval $[a,b]$. Assume that for all $x_1, x_2 \in [a,b]$*

$$-M \leq Re\int_{x_1}^{x_2} \lambda(s)\,ds \leq M, \tag{2.5}$$

where M is a fixed number. Let the number ρ, defined below, satisfy

$$\rho = \int_a^b |r(t)|\,dt < \infty. \tag{2.6}$$

Then (2.1) possesses two linearly independent solutions

$$y_1 = [(\theta + i\psi)(1+p_{11}) - i(\theta - i\psi)p_{21}]\sqrt[4]{\phi}^{-1}\exp\int^x \sqrt{\phi + (\frac{1}{4}\frac{\phi'}{\phi})^2}\,dt \tag{2.7}$$

$$y_2 = [(\theta + i\psi)p_{12} - i(\theta - i\psi)(1+p_{22})]\sqrt[4]{\phi}^{-1}\exp -\int^x \sqrt{\phi + (\frac{1}{4}\frac{\phi'}{\phi})^2}\,dt \tag{2.8}$$

and their derivatives are given by

$$y_1' = [(\theta - i\psi)(1+p_{11}) + i(\theta + i\psi)p_{21}]\sqrt[4]{\phi}^{-1}\exp\int^x \sqrt{\phi + (\frac{1}{4}\frac{\phi'}{\phi})^2}\,dt \tag{2.9}$$

$$y_2' = [(\theta - i\psi)p_{12} + i(\theta + i\psi)(1+p_{22})]\sqrt[4]{\phi}^{-1}\exp -\int^x \sqrt{\phi + (\frac{1}{4}\frac{\phi'}{\phi})^2}\,dt \tag{2.10}$$

where the mappings p_{11}, p_{12}, p_{21} and p_{22} are "perturbations" which can be calculated from certain integral equations given in [3] and

$$\theta = m + m^{-1}, \tag{2.11}$$

$$\psi = m - m^{-1}, \tag{2.12}$$

$$m = [\frac{1-il}{1+il}]^{1/4}. \tag{2.13}$$

In the case $l^2(b) = -1$, the solution given in [4] has different representations with respect to (1): b is a singular regular point of $\phi(x)$ and (2): b is a regular point of $\phi(x)$.

Theorem 2.2 *Let $\phi(x)$ be meromorphic on $[a, b]$. Also assume that*

$$l^2(b^-) = -1.$$

Then, (2.1) is nonoscillatory at b^-, and when b is a singular regular point for (2.1) we have

$$y_1 = (mq)^{-1}(1-g)t_{21}(1+q_{22}+\frac{e_1q_{21}}{t_{21}}+\frac{(t_{21}q_{12}+e_1(1+q_{11})(1+\bar{g})}{t_{21}(1-g)}, \tag{2.14}$$

$$y_2 = (mq)^{-1}(1-g)e_2(1+q_{22}+\frac{q_{12}(1+\bar{g})}{1-g}, \tag{2.15}$$

$$y_1' = -m^{-1}q(1+g)t_{21}(1+q_{22}+\frac{e_1q_{21}}{t_{21}}-\frac{(t_{21}q_{12}+e_1(1+q_{11}))(1-\bar{g})}{(1+g)t_{21}}, \tag{2.16}$$

$$y_2' = -m^{-1}q(1+g)e_2(1+q_{22}+\frac{q_{12}(1-\bar{g})}{1+g}). \tag{2.17}$$

In this case $|t_{21}(b^-)| = \infty$.

If b is a regular point, then

$$y_1 = (mq)^{-1}(1+\bar{g})e_1(1+q_{11}+\frac{q_{21}(1-g)}{1+\bar{g}}+\frac{t_{21}q_{12}}{e_1}+\frac{t_{21}(1+q_{22})(1-g)}{e_1(1+\bar{g})}), \tag{2.18}$$

$$y_2 = (mq)^{-1}(1-g)e_2(1+q_{22}+\frac{q_{12}(1+\bar{g})}{1-g}), \tag{2.19}$$

$$y_1' = m^{-1}q(1-\bar{g})e_1(1+q_{11}-\frac{q_{21}(1+g)}{1-\bar{g}}+\frac{t_{21}q_{12}}{e_1}-\frac{t_{21}(1+q_{22})(1+g)}{e_1(1-\bar{g})}), \tag{2.20}$$

$$y_2' = -m^{-1}q(1+g)e_2(1+q_{22}+\frac{q_{12}(1-\bar{g})}{1+g}), \tag{2.21}$$

where the mappings q_{11}, q_{12}, q_{21} and q_{22} are "perturbations" which can be determined from a set of integral equations given in [4] and the functions in (2.14) to (2.21) are given as below.

$$q = \phi^{\frac{1}{4}}(x), \tag{2.22}$$

$$g = \frac{\lambda - d}{c}, d_1 = q^2, c_1 = \frac{q'}{q}, \tag{2.23}$$

$$m = (1+g\bar{g})^{\frac{1}{2}}, \tag{2.24}$$

$$e_1 = \exp\int_b^x \lambda_1(s)\,ds, e_2 = \exp\int_b^x \lambda_2(s)\,ds, \tag{2.25}$$

$$\lambda_1 = \lambda - (g\bar{g}' - mm')m^{-2}, \quad \lambda_2 = -\lambda - (\bar{g}g' - mm')m^{-2}, \tag{2.26}$$

$$t_{21} = \delta(x, x_0, x)\exp\int_b^x \lambda_1(s)\,ds, \tag{2.27}$$

$$\delta(x, s_1, s_2) = \exp \int_{s_1}^{x} r_{21}(s) \exp \int_{s_2}^{s} (\lambda_1 - \lambda_2)\, d\eta\, ds, \tag{2.28}$$

$$r_{21} = \frac{\phi'}{2\phi} - m^{-2}\bar{g}'. \tag{2.29}$$

Note that equations (2.14) to (2.17) as well as equations (2.18) to (2.21) follow from one and the same matrix formula representation in [4].

3. The various cases considered

The original equation we are going to discuss is given ass

$$\frac{d}{dx}\left(\frac{1-x^2}{\sigma^2-x^2}\frac{dy}{dx}\right) + \left\{\frac{1}{\sigma^2-x^2}\left[\frac{\hat{m}(\sigma^2+x^2)}{\sigma(\sigma^2-x^2)} - \hat{m}^2\right] + \epsilon\right\} y = 0, \tag{3.1}$$

where $\sigma, \hat{m}$ and ϵ represent some parameters.

With a straight forward transformation of equation (3.1), we reach a general second order linear differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0, \tag{3.2}$$

where

$$a(x) = \frac{1-x^2}{\sigma^2-x^2}, b(x) = \frac{2x(1-\sigma^2)}{(\sigma^2-x^2)^2}, c(x) = \frac{1}{(\sigma^2-x^2)}\left[\frac{\hat{m}(\sigma^2+x^2)}{\sigma(\sigma^2-x^2)} - \hat{m}^2\right] + \epsilon. \tag{3.3}$$

The linear transformation

$$y(x) = u(x)z(x) \tag{3.4}$$

with

$$u(x) = \frac{\sqrt{\sigma^2-x^2}}{\sqrt{1-x^2}}$$

takes the differential equation (3.2) into the canonical form

$$z''(x) = \phi(x)z(x) \tag{3.5}$$

here

$$\phi(x) = \frac{\sigma(1-\sigma^2)(\sigma^2+2x^2-3x^4) + \sigma\hat{m}^2(1-x^2)(\sigma^2-x^2)^2}{\sigma(1-x^2)^2(\sigma^2-x^2)^2} - \frac{\hat{m}(1-x^2)(\sigma^4-x^4) + \sigma\epsilon(\sigma^2-x^2)^3(1-x^2)}{\sigma(1-x^2)^2(\sigma^2-x^2)^2} \tag{3.6}$$

which has singular regular points at $x = \pm 1$ and $x = \pm\sigma$.

We remark that the Liouville-Green (or WKB) approximations can not be applied here and the Frobenius' method, although applicable, is inconvenient to use. This is so because for σ close to 1, the "perturbation" power series obtained from the Frobenius' method will have a very small radius of convergence.

Since $\phi(x)$ is a function of x^2, we consider only the asymptotic behaviors of the solutions at $x=1$, and $x=\sigma$ for $\sigma>0$.

Our first goal is to calculate the comparability index. A straight forward calculation reveals that

$$
\begin{aligned}
l(x) &= \frac{\phi'(x)}{4\phi^{\frac{3}{2}}(x)}\\
&= \sqrt{\sigma}\frac{\sigma(1-\sigma^2)x(2\sigma^2+x^2+\sigma^4+3x^6-3x^4-4x^2-\sigma^2)-\hat{m}\sigma x^2(1-x^2)^2(\sigma^2-x^2)}{\{\sigma(1-\sigma^2)(\sigma^2+2x^2-3x^4)+(1-x^2)[\sigma\hat{m}^2(\sigma^2-x^2)^2-\hat{m}(\sigma^4-x^4)+\sigma\epsilon(\sigma^2-x^2)^3]\}^{\frac{3}{2}}}\\
&+ \frac{(x/2)(1-x^2)(\sigma^2-x^2)[\sigma\hat{m}^2(\sigma^2-x^2)-\hat{m}(\sigma^2+x^2)](\epsilon\sigma/2)(\sigma^2-1)x(1-x^2)(\sigma^2-x^2)^3}{\{\sigma(1-\sigma^2)(\sigma^2+2x^2-3x^4)+(1-x^2)[\sigma\hat{m}^2(\sigma^2-x^2)^2-\hat{m}(\sigma^4-x^4)+\sigma\epsilon(\sigma^2-x^2)^3]\}^{\frac{3}{2}}}.
\end{aligned}
\tag{3.7}
$$

We can then generate a table for $l(x)$ as $x\to 1$ and $x\to\sigma(\sigma\neq 1)$ as shown below:

	$l(x)$	$l^2(x)$
$x=1(x\neq\sigma)$	$-i$	-1
$x=\sigma$	$1/\sqrt{3}$	$1/3$

Because of the above limits, we distinguish among the following cases.

Case 1. $x\to 1$ with $\sigma\neq 1$;

Case 2. $x\to 1$ with $\sigma=1$;

Case 3. $x\to\sigma$ with $\sigma\neq 1$.

In the following sections, we are going to utilize the two theorems of section 2 in order to handle the variety of cases summarized above.

4. Application of theorem 2.1 for $\sigma\neq 1$

Now we will apply Theorem 2.1 and Theorem 2.2 to derive the general solutions for (3.1) with the consideration of the three cases mentioned at the end of section 3. For the sake of exposition, we will only calculate the leading asymptotic terms and set the perturbation terms p_{jk} and $q_{jk}, j,k=1,2$ in our formulas to be zero.

We start with Case 3. From the table in the last section, we know that as $x\to\sigma$ and $\sigma\neq 1, l^2(x)\to 1/3$. Therefore Theorem 2.1 is applicable.

As $x\to\sigma$, the $\phi(x)$ in (3.6) behaves like

$$
\begin{aligned}
\phi(x) &= \frac{3(1-\sigma^2)x^2}{(1-x^2)(\sigma^2-x^2)^2}\{1+[\frac{1}{3x^2(1-x^2)}-\frac{\hat{m}(\sigma^2+x^2)}{3\sigma(1-\sigma^2)x^2}](\sigma^2-x^2)\\
&+ \frac{\hat{m}^2}{3(1-\sigma^2)x^2}(\sigma^2-x^2)^2+\frac{\epsilon}{3(1-\sigma^2)x^2}(\sigma^2-x^2)^3]\}\\
&\sim \frac{3(1-\sigma^2)x^2}{(1-x^2)(\sigma^2-x^2)^2}.
\end{aligned}
\tag{4.1}
$$

In order to use Theorem 2.1, we need certain limitary values of mappings as $x\to\sigma$. According to equations (2.13), (2.12) and (2.11) we have

$$m(\sigma)=(\frac{\sqrt{3}-i}{\sqrt{3}+i})^{\frac{1}{4}}, \tag{4.2}$$

$$\psi(\sigma) = (\frac{\sqrt{3}-i}{\sqrt{3}+i})^{\frac{1}{4}} - (\frac{\sqrt{3}+i}{\sqrt{3}-i})^{\frac{1}{4}} = \frac{(\sqrt{3}-i)^{\frac{1}{2}} - (\sqrt{3}+i)^{\frac{1}{2}}}{\sqrt{2}}, \tag{4.3}$$

and

$$\theta(\sigma) = (\frac{\sqrt{3}-i}{\sqrt{3}+i})^{\frac{1}{4}} + (\frac{\sqrt{3}+i}{\sqrt{3}-i})^{\frac{1}{4}} = \frac{(\sqrt{3}-i)^{\frac{1}{2}} + (\sqrt{3}+i)^{\frac{1}{2}}}{\sqrt{2}}. \tag{4.4}$$

By plugging (4.3) and (4.4) into (2.7), (2.8), (2.9) and (2.10), equation (3.5) has the general solution

$$\begin{aligned} z_1(x) &= (\theta+\psi)(\phi(x))^{-\frac{1}{4}} \exp(\int_{x_0}^{x} (\phi(s)(1+l^2(s)))^{\frac{1}{2}}\, ds \\ &\sim \frac{((\sqrt{3}-i)^{\frac{1}{2}}(1+i) + (\sqrt{3}+i)^{\frac{1}{2}}(1-i)(\sigma^2-x^2)^{\frac{1}{2}}(1-x^2)^{\frac{1}{4}}}{3^{\frac{1}{4}}\sqrt{2x}(1-\sigma^2)^{\frac{1}{4}}(\sigma-x)}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} z_2(x) &= -i(\theta-i\psi)(\phi(x))^{-\frac{1}{4}} \exp(-\int_{x_0}^{x} (\phi(s)(1+l^2(s)))^{\frac{1}{2}}\, ds \\ &\sim \frac{((\sqrt{3}+i)^{\frac{1}{2}}(1-i) + (\sqrt{3}+i)^{\frac{1}{2}}(1+i)(\sigma^2-x^2)^{\frac{1}{2}}(1-x^2)^{\frac{1}{4}}(\sigma-x)}{3^{\frac{1}{4}}\sqrt{2x}(1-\sigma^2)^{\frac{1}{4}}}, \end{aligned} \tag{4.6}$$

with derivatives

$$\begin{aligned} z_1'(x) &= (\theta-i\psi)(\phi(x))^{\frac{1}{4}} \exp(\int^{x} (\phi(s)(1+l^2(s)))^{\frac{1}{2}}\, ds) \\ &\sim \frac{((\sqrt{3}-i)^{\frac{1}{2}}(1-i) + (\sqrt{3}+i)^{\frac{1}{2}}(1+i))3^{\frac{1}{4}}\sqrt{x}(1-\sigma^2)^{\frac{1}{4}}}{\sqrt{2}(\sigma^2-x^2)^{\frac{1}{2}}(1-x^2)^{\frac{1}{4}}(\sigma-x)}, \end{aligned} \tag{4.7}$$

$$\begin{aligned} z_2'(x) &= i(\theta+i\psi)(\phi(x))^{\frac{1}{4}} \exp(-\int^{x} (\phi(s)(1+l^2(s)))^{\frac{1}{2}}\, ds) \\ &\sim \frac{((\sqrt{3}+i)^{\frac{1}{2}}(1+i) - (\sqrt{3}-i)^{\frac{1}{2}}(1-i))3^{\frac{1}{4}}\sqrt{x}(1-\sigma^2)^{\frac{1}{4}}(\sigma-x)}{\sqrt{2}(\sigma^2-x^2)^{\frac{1}{2}}(1-x^2)^{\frac{1}{4}}}, \end{aligned} \tag{4.8}$$

From equation (3.4), we can transform x into y. That is, the general solutions of equation (3.1) are given as

$$y_1(x) = u(x)z_1(x) = \frac{(\sigma+x)((\sqrt{3}-i)^{\frac{1}{2}}(1+i) + (\sqrt{3}+i)^{\frac{1}{2}}(1-i))}{3^{\frac{1}{4}}\sqrt{2x}(1+x)^{\frac{1}{4}}(1-x)^{\frac{1}{4}}(1-\sigma^2)^{\frac{1}{4}}}, \tag{4.9}$$

$$y_2(x) = u(x)z_2(x) = \frac{(\sigma+x)((\sqrt{3}+i)^{\frac{1}{2}}(1-i) - (\sqrt{3}-i)^{\frac{1}{2}}(1+i))(\sigma-x)^2}{3^{\frac{1}{4}}\sqrt{2x}(1+x)^{\frac{1}{4}}(1-x)^{\frac{1}{4}}(1-\sigma^2)^{\frac{1}{4}}}. \tag{4.10}$$

5. Application of Theorem 2.2 for $\sigma \neq 1$

We now consider the nature of solutions of (3.1) as $x \to 1$. From (3.6), $\phi(x)$ behaves like

$$\begin{aligned}\phi(x) &= \frac{(1-\sigma^2)}{(1-x^2)^2(\sigma^2-x^2)}\{1+(1-x^2)[\frac{3x^2}{(\sigma^2-x^2)} - \frac{\hat{m}(\sigma^2+x^2)}{\sigma(1-\sigma^2)} + \frac{\hat{m}^2(\sigma^2-x^2)}{1-\sigma^2} \\ &+ \frac{\epsilon(\sigma^2-x^2)^2}{(1-\sigma^2)}]\} \quad \text{as} \quad x \to 1. \end{aligned} \tag{5.1}$$

Since $l^2(x) \to -1$ in this case, we can assume that asymptotically

$$l^2(x) = -1 + \alpha_1(1-x^2) + \alpha_2(1-x^2)^2 + \cdots, \tag{5.2}$$

here α_1, α and all those coefficients for the higher order terms can be calculated by means of comparison of the coefficients of $(1-x^2)$ of all powers in (5.2) with the expression from (3.7) with $x \to 1$. We give only α_1 here since it is the dominant coefficient.

$$\begin{aligned}\alpha_1 &= \frac{(1-\sigma^2)[\sigma\hat{m}^2(\sigma^2-1) - \hat{m}(\sigma^2+1) + \epsilon\sigma(1-\sigma^2)^2]}{\sigma(1-\sigma^2)^2} \\ &+ \frac{3[\sigma\hat{m}^2(\sigma^2-1) - \hat{m}(\sigma^2+1) + \sigma\epsilon(\sigma^2-1)^2]}{\sigma(1-\sigma^2)^2}.\end{aligned}$$

According to (5.2), we calculate the following mappings which will be used to develop the general solution. From equation (2.3), we get

$$\begin{aligned}\lambda(x) &= [\phi(x)(1+l^2(x))]^{\frac{1}{2}} \\ &= [(1-\sigma^2)(\alpha_1(1-x^2) + \alpha_2(1-x^2)^2 + \cdots)(1-x^2)^2(\sigma^2-x^2)]^{\frac{1}{2}} \\ &\sim \frac{\alpha_1}{2(x-1)} \quad \text{as} \quad x \to 1\end{aligned} \tag{5.3}$$

and from equation (2.23) and (2.24), we have

$$d_1(x) = \phi^{\frac{1}{2}}(x) \sim \frac{1}{(1-x^2)}\sqrt{\frac{1-\sigma^2}{\sigma^2-x^2}} = \frac{i}{2(1-x)} \quad \text{as} \quad x \to 1, \tag{5.4}$$

$$c_1(x) = l(x)d_1(x) \sim \frac{1}{2(1-x)} \quad \text{as} \quad x \to 1, \tag{5.5}$$

$$g(x) = \frac{\lambda(x) - d_1(x)}{c_1(x)} \sim -i \quad \text{as} \quad x \to 1, \tag{5.6}$$

$$m(x) = (1+g\bar{g})^{\frac{1}{2}} \quad \text{as} \quad x \to 1. \tag{5.7}$$

By equation (2.26), we have

$$\begin{aligned}&\lambda_1(x) = \lambda(x) - (g\bar{g} - mm')m^{-2} \sim \lambda(x) - \frac{(g\bar{g}' - g'\bar{g})m^{-2}}{2} \\ \sim\ &\lambda(x) - \frac{-ig' - ig')m^{-2}}{2} \sim [\frac{\alpha_1}{2(x-1)}]^{\frac{1}{2}} + i\frac{g'}{2} \quad \text{as} \quad x \to 1\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}
\lambda_2(x) &= -\lambda(x) - (\bar{g}g' - mm')m^{-2} \\
\sim \quad & -\lambda(x) - \frac{(\bar{g}g' - \bar{g}'g)m^{-2}}{2} \\
\sim \quad & -\lambda(x) - \frac{(ig' + ig')m^{-2}}{2} \\
\sim \quad & -\lambda_1(x) \quad \text{as} \quad x \to 1.
\end{aligned} \tag{5.9}$$

We also have, from equation (2.25)

$$\begin{aligned}
e_1(x) &= \exp(\int_{x_0}^{x} \lambda_1(s)\, ds) \\
\sim \quad & \exp(-\sqrt{2\alpha_1(x_0 - 1)} - i\frac{g(x_0)}{2} + \frac{1}{2}) \quad \text{as} \quad \alpha \to 1
\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}
e_2(x) &= \exp(\int_{x_0}^{x} \lambda_2(s)\, ds) \\
&\sim \exp(\sqrt{2\alpha_1(x_0 - 1)} + i\frac{g(x_0)}{2} - \frac{1}{2}) \qquad \text{as} \qquad x \to 1,
\end{aligned} \tag{5.11}$$

where x_0 is a point in the neighborhood of $x = 1$, which is arbitrary and is equal to one for convenience. If we choose $x_0 = 1$, then we have

$$e_1 \sim 1 \qquad \text{and} \qquad e_2 \sim 1.$$

Finally by equation (2.27) we have

$$\begin{aligned}
t_{21}(x) &= e_1\delta(x, x_0, x) \\
&\sim \int_{x_0}^{x} (\frac{\phi'(s)}{2\phi(s)} - m^{-2}(s)\bar{g}'(s)) \exp(2\int_{x}^{s} \lambda_1(t)\, dt)\, ds)e_1 \\
&\sim \frac{1}{2}(\ln \frac{\phi(x)}{\phi(x_0)} + i + g(x_0)) \qquad \text{as} \qquad x \to 1
\end{aligned} \tag{5.12}$$

where x_0 could be any number different than 1.

Hence by Theorem 2.2, as $x \to 1$, we have two solutions z_1 and z_2 and their derivatives as below

$$\begin{aligned}
z_1(x) &= (mq)^{-1}\{e_1(1 + \bar{g}) + t_{21}(1 - g)\} \\
&\sim \frac{(1 - x^2)^{\frac{1}{2}}(\sigma^2 - x^2)^{\frac{1}{4}}(1 + i)}{\sqrt{2}(1 - \sigma^2)^{\frac{1}{4}}}[1 + \frac{1}{2}(\ln \frac{\phi(x)}{\phi(x_0)} + i + g(x_0))],
\end{aligned} \tag{5.13}$$

$$\begin{aligned}
z_2(x) &= (mq)^{-1}(1 - g) \\
&\sim \frac{(1 - x^2)^{\frac{1}{2}}(\sigma^2 - x^2)^{\frac{1}{4}}(1 + i)}{\sqrt{2}(1 - \sigma^2)^{\frac{1}{4}}}
\end{aligned} \tag{5.14}$$

and

$$z_1'(x) = m^{-1}q\{e_1(1-g) + t_{21}(1+g)\}$$
$$\sim \frac{(1-i)}{2(x-1)^{\frac{1}{2}}}[1 + \frac{1}{2}(\ln\frac{\phi(x)}{\phi(x_0)} + i + g(x_0))], \tag{5.15}$$

$$z_2'(x) = -m^{-1}qe_2(1+g) \sim -\frac{(1-i)}{2(x-1)^{\frac{1}{2}}}. \tag{5.16}$$

Again from (3.6), we have the general solutions of (3.1) as x approaches to 1.

$$y_1(x) = u(x)z_1(x)$$
$$\sim \frac{(\sigma^2 - x^2)^{\frac{3}{4}}(1+i)}{\sqrt{2}(1-\sigma^2)^{\frac{1}{4}}}[1 + \frac{1}{2}(\ln\frac{\phi(x)}{\phi(x_0)} + i + g(x_0))], \tag{5.17}$$

$$y_2(x) = u(x)z_2(x) \sim \frac{(\sigma^2 - x^2)^{\frac{3}{4}}(1+i)}{\sqrt{2}(1-\sigma^2)^{\frac{1}{4}}} \tag{5.18}$$

As we can see, y_1 and y_2 are bounded as x goes to 1.

6. The application of Theorem 2.1 and Theorem 2.2 for $\sigma = 1$

We now go ahead to consider the general solution in the case $x \to 1$ with $\sigma = 1$. For this case, equation (3.6) reduces to

$$\phi(x) = \frac{m(1-x^2)^3 + \epsilon(1-x^2)^4 - m(1-x^2)(1-x^4)}{(1-x^4)^4}. \tag{6.1}$$

As $x \to 1$, $\phi(x)$ is approximated as follows

$$\phi(x) \sim \frac{\hat{m}^2}{(1-x^2)} - \frac{\hat{m}(1+x^2)}{(1-x^2)^2} + \epsilon \sim -\frac{\hat{m}(1+x^2)}{(1-x^2)^2} \sim -\frac{\hat{m}}{2(1-x)^2}. \tag{6.2}$$

By plugging $\sigma = 1$ into (3.7), we get

$$l(x) = \frac{(x/2)(1-x^2)^3[\hat{m}^2(1-x^2) - \hat{m}(1+x^2)] - \hat{m}x(1-x^2)^3}{[\hat{m}^2(1-x^2)^3 - \hat{m}(1-x^2)^2(1+x^2) + \epsilon(1-x^2)^4]^{\frac{3}{2}}}. \tag{6.3}$$

Hence for $\sigma = 1$

$$l^2(x) \sim (-\frac{2}{(-8\hat{m})^{\frac{1}{2}}})^2 = -\frac{1}{2m} \quad \text{as} \quad x \to 1, \tag{6.4}$$

$\hat{m}$ could be any value. But only two kinds of values make differences in the approximation. So we consider the following two cases:

Case 1: $\hat{m} = \frac{1}{2}$, that is $l^2(x) = -1$.

We again assume that as $x \to 1$

$$l^2(x) = -1 + \alpha_1(1-x^2) + \alpha_2(1-x^2)^2 + \cdots, \tag{6.5}$$

where α_1, α_2 and so on can be determined by calculating the coefficients of (6.5) and (6.3). For $\hat{m} = \frac{1}{2}$, our manipulation shows that

$$\alpha_1 = \frac{x^2}{12(1+x^2)^2} - \frac{\hat{m}}{1+x^2} = -\frac{11}{48} \quad \text{as} \quad x \to 1.$$

With the series expansion of $l^2(x)$ in hand, the mapping defined in equation (2.3) has a neat form as below:

$$\begin{aligned}
\lambda(x) &= [\phi(x)(1+l^2(x))]^{\frac{1}{2}} \\
&= \sqrt{-\frac{\hat{m}(1+x^2)(\alpha_1(1-x^2)+\alpha_2(1-x^2)^2+\cdots)}{(1-x^2)^2}} \\
&\sim \sqrt{\frac{\alpha_1}{2(x-1)}} \quad \text{as} \quad x \to 1.
\end{aligned} \tag{6.6}$$

Again we calculate the following functions according to Theorem 2.2 with the asymptotic approximation (6.5) and (6.6), with the same way as we did in section 5.

$$d_1(x) = \phi^{\frac{1}{2}}(x) \sim \sqrt{-\frac{\hat{m}(1+x^2)}{(1-x^2)^2}} \sim \frac{i}{2(1-x)} \quad \text{as} \quad x \to 1, \tag{6.7}$$

$$\begin{aligned}
c_1(x) &= l(x)d_1(x) \sim \frac{-(x/2)\hat{m}(1+x^2) - \hat{m}x}{\sqrt{(-\hat{m}(1+x^2))^3}} \frac{\sqrt{-\hat{m}(1+x^2)}}{(1-x^2)} \\
&\sim \frac{1}{2(1-x)} \quad \text{as} \quad x \to 1
\end{aligned} \tag{6.8}$$

$$q(x) = \phi(x)^{\frac{1}{4}} \sim \frac{(-\hat{m}(1+x^2))^{\frac{1}{4}}}{\sqrt{(1-x^2)}} \sim \frac{(-1)^{\frac{1}{4}}}{\sqrt{2(1-x)}} \quad \text{as} \quad x \to 1, \tag{6.9}$$

$$g(x) = \frac{\lambda(x) - d_1(x)}{c_1(x)} \sim -i \quad \text{as} \quad x \to 1, \tag{6.10}$$

$$m(x) = (1 + g(x)\bar{g}(x))^{\frac{1}{2}} = \sqrt{2} \quad \text{as} \quad x \to 1, \tag{6.11}$$

$$\begin{aligned}
\lambda_1(x) &= \lambda(x) - (g\bar{g}' - mm')m^{-2} \sim \lambda(x) - \frac{(g\bar{g}' - g'\bar{g})m^{-2}}{2} \\
&\sim \lambda(x) - \frac{(-ig' - ig')m^{-2}}{2} \sim \sqrt{\frac{\alpha_1}{2(x-1)}} + \frac{ig'}{2} \quad \text{as} \quad x \to 1,
\end{aligned} \tag{6.12}$$

$$\begin{aligned}
\lambda_2(x) &= -\lambda(x) - (g'\bar{g} - mm')m^{-2} \sim -\lambda(x) - \frac{(g'\bar{g} - g\bar{g}')m^{-2}}{2} \\
&\sim -\lambda(x) - \frac{(ig' + ig')m^{-2}}{2} \sim -\sqrt{\frac{\alpha_1}{2(x-1)}} - \frac{ig'}{2} \\
&= -\lambda_1(x) \quad \text{as} \quad x \to 1.
\end{aligned} \tag{6.13}$$

We get from (2.25) that

$$\begin{aligned} e_1(x) &= \exp\left(\int_{x_0}^{x} \lambda_1(s)\,ds\right) \\ &\sim \exp\left(-\sqrt{2\alpha_1(x_0-1)} - \frac{ig(x_0)}{2} + \frac{1}{2}\right) \qquad \text{as} \quad x \to 1, \end{aligned} \tag{6.14}$$

$$\begin{aligned} e_1(x) &= \exp\left(\int_{x_0}^{x} \lambda_2(s)\,ds\right) \\ &\sim \exp\left(\sqrt{2\alpha_1(x_0-1)} + \frac{ig(x_0)}{2} - \frac{1}{2}\right) \qquad \text{as} \quad x \to 1, \end{aligned} \tag{6.15}$$

where x_0 is any value in the neighborhood of $x = 1$. For the purpose of brevity we let x_0 be 1. Hence we have

$$e_1 \sim 1 \qquad \text{and} \qquad e_2 \sim 1.$$

Also as $x \to 1$

$$\begin{aligned} t_{21}(x) &= \delta(x, x_0, x)e_1(x) \sim \int_{x_0}^{x}\left[\frac{\phi'(s)}{2\phi(s)} - \frac{\overline{g'(s)}}{m^2(s)}\right]\exp\left(2\int_{x}^{s}\lambda_1(\eta)\,d\eta\right) ds\, e_1(x) \\ &= \left[\frac{1}{2}\ln\frac{\phi(x)}{\phi(x_0)} + \frac{1}{2}(i + g(x_0))\right] \end{aligned} \tag{6.16}$$

The above functions enable us to invoke Theorem 2.2, we have two linearly independent solutions

$$\begin{aligned} z_1(x) &= (mq)^{-1}\{e_1(1+\bar{g}) + t_{21}(1-g)\} \\ &\sim \frac{\sqrt{(1-x^2)}(\sigma^2 - x^2)^{\frac{1}{4}}(1+i)}{\sqrt{2}(1-\sigma^2)^{\frac{1}{4}}}\left[1 + \frac{1}{2}\ln\frac{\phi(x)}{\phi(x_0)} + \frac{1}{2}(i + g(x_0))\right], \end{aligned} \tag{6.17}$$

$$\begin{aligned} z_2(x) &= (mq)^{-1}e_2(1-g) \\ &\sim \frac{\sqrt{(1-x^2)}(\sigma^2 - x^2)^{\frac{1}{4}}(1+i)}{\sqrt{2}(1-\sigma^2)^{\frac{1}{4}}} \end{aligned} \tag{6.18}$$

and their derivatives,

$$\begin{aligned} z_1'(x) &= m^{-1}\{e_1(1-g) + t_{21}(1+g)\} \\ &\sim \frac{(1-i)}{2\sqrt{x-1}}\left[1 + \frac{1}{2}\left(\ln\frac{\phi(x)}{\phi(x_0)} + i + g(x_0)\right)\right], \end{aligned} \tag{6.19}$$

$$z_2'(x) = -m^{-1}qe_2(1+g) \sim -\frac{(1-i)}{2\sqrt{x-1}}. \tag{6.20}$$

y_1, y_2, y_1' and y_2' can be obtained using equation (3.4). We calculated for y_1 and y_2 as

$$\begin{aligned} y_1(x) &= u(x)z_1(x) \\ &\sim \frac{(\sigma^2 - x^2)^{\frac{3}{4}}(1+i)}{\sqrt{2}(1-\sigma^2)^{\frac{1}{4}}}\left[1 + \frac{1}{2}\left(\ln\frac{\phi(x)}{\phi(x_0)} + i + g(x_0)\right)\right] \end{aligned} \tag{6.21}$$

and

$$y_2(x) = u(x)z_2(x) \sim \frac{(\sigma^2 - x^2)^{\frac{3}{4}}(1+i)}{\sqrt{2}(1-\sigma^2)^{\frac{1}{4}}}. \tag{6.22}$$

We see again that y_1 and y_2 are bounded at the limit.

Let us now examine the other case.

Case 2: $\hat{m} \neq \frac{1}{2}$.

$$l(x) \sim \frac{i}{\sqrt{2\hat{m}}} \qquad \text{as} \quad x \to 1. \tag{6.23}$$

Since $\hat{m} \neq \frac{1}{2}$, so $l(x) \neq -1$. Theorem 2.1 is applicable.

$$m(x) = \left(\frac{1 - il(x)}{1 + il(x)}\right)^{\frac{1}{4}} \sim \left(\frac{\sqrt{2}\hat{m}^{\frac{1}{2}} + 1}{\sqrt{2}\hat{m}^{\frac{1}{2}} - 1}\right)^{\frac{1}{4}} \qquad \text{as} \quad x \to 1, \tag{6.24}$$

$$\psi(x) = m - m^{-1} \sim \frac{\sqrt{\sqrt{2\hat{m}} + 1} - \sqrt{\sqrt{2\hat{m}} - 1}}{(2\hat{m} - 1)^{\frac{1}{4}}} \qquad \text{as} \quad x \to 1, \tag{6.25}$$

$$\theta(x) = m + m^{-1} \sim \frac{\sqrt{\sqrt{2\hat{m}} + 1} + \sqrt{\sqrt{2\hat{m}} - 1}}{(2\hat{m} - 1)^{\frac{1}{4}}} \qquad \text{as} \quad x \to 1. \tag{6.26}$$

Therefore by applying Theorem 2.1 we obtain the asymptotic form of solutions.

$$z_1(x) = (\theta + \psi)\phi^{-\frac{1}{4}} \exp(\int_{x_0}^{x} (\phi(s)(1 + l^2(s)))^{\frac{1}{2}}\, ds)$$

$$\sim \frac{\sqrt{\sqrt{2\hat{m}} + 1}(1+i) + \sqrt{\sqrt{2\hat{m}} - 1}(1-i)}{(2\hat{m} - 1)^{\frac{1}{4}}} \frac{2^{\frac{1}{4}}\sqrt{1-x}}{(-\hat{m}^{\frac{1}{4}}} \exp[\frac{1}{2}\sqrt{1 - 2\hat{m}} \ln(\frac{2}{1-x})], \tag{6.27}$$

$$z_2(x) = -i(\theta - i\psi)\phi^{-\frac{1}{4}} \exp(-\int_{x_0}^{x} (\phi(s)(1 + l^2(s)))^{\frac{1}{2}}\, ds)$$

$$\sim \frac{\sqrt{\sqrt{2\hat{m}} - 1}(1-i) - \sqrt{\sqrt{2\hat{m}} + 1}(1+i)}{(2\hat{m} - 1)^{\frac{1}{4}}} \frac{2^{\frac{1}{4}}\sqrt{1-x}}{(-\hat{m})^{\frac{1}{4}}} \exp[-\frac{1}{2}\sqrt{1 - 2\hat{m}} \ln(\frac{2}{1-x})] \tag{6.28}$$

From Theorem 2.1, their derivatives are

$$z_1'(x) = (\theta - i\psi)\phi^{\frac{1}{4}} \exp(\int_{x_0}^{x} (\phi(s)(1 + l^2(s)))^{\frac{1}{2}}\, ds)$$

$$\sim \frac{\sqrt{\sqrt{2\hat{m}} + 1}(1-i) + \sqrt{\sqrt{2\hat{m}} - 1}(1+i)}{(2\hat{m} - 1)^{\frac{1}{4}}\sqrt{1-x}} \frac{(-\hat{m})^{\frac{1}{4}}}{2^{\frac{1}{4}}} \exp[\frac{1}{2}\sqrt{1 - 2\hat{m}} \ln(\frac{2}{1-x})], \tag{6.29}$$

$$z_2'(x) = i(\theta + i\psi)\phi^{\frac{1}{4}} \exp(-\int_{x_0}^{x} (\phi(s)(1 + l^2(s)))^{\frac{1}{2}}\, ds)$$

$$\sim \frac{\sqrt{\sqrt{2\hat{m}} - 1}(1+i) - \sqrt{\sqrt{2\hat{m}} + 1}(1-i)}{(2\hat{m} - 1)^{\frac{1}{4}}\sqrt{1-x}} \frac{(-\hat{m})^{\frac{1}{4}}}{2^{\frac{1}{4}}} \exp[-\frac{1}{2}\sqrt{1 - 2\hat{m}} \ln(\frac{2}{1-x})]. \tag{6.30}$$

By utilizing equation (3.4), we obtain the corresponding approximations to y_1, y_2, y_1' and y_2'. Here we give only y_1 and y_2,

$$\begin{aligned} y_1(x) &= u(x)z_1(x) \\ &\sim \frac{\sqrt{\sqrt{2\hat{m}}+1}(1+i)+\sqrt{\sqrt{2\hat{m}}-1}(1-i)}{(2\hat{m}-1)^{\frac{1}{4}}} \frac{\sqrt{\sigma^2-x^2}}{(2(-\hat{m}))^{\frac{1}{4}}} \exp[\frac{1}{2}\sqrt{1-2\hat{m}}\ln(\frac{2}{1-x})], \end{aligned} \tag{6.31}$$

$$\begin{aligned} y_2(x) &= u(x)z_2(x) \\ &\sim \frac{\sqrt{\sqrt{2\hat{m}}-1}(1-i)-\sqrt{\sqrt{2\hat{m}}+1}(1+i)}{(2\hat{m}-1)^{\frac{1}{4}}} \frac{\sqrt{\sigma^2-x^2}}{(2(-\hat{m}))^{\frac{1}{4}}} \exp[-\frac{1}{2}\sqrt{1-2\hat{m}}\ln(\frac{2}{1-x})]. \end{aligned} \tag{6.32}$$

7. Concluding remarks

The asymptotic solutions at the singular points as well as regular points of second order ODE's can be treated with two packages of approximations. We demonstrated this for an equation with various parameters which involved the consideration of many cases. Each of these cases was discussed separately, with each invoking either Theorem 2.1 or Theorem 2.2. Our unified approach could have an edge on the conventional traditional methods. Some of the benefits have already been reaped in problems of wave propagation.

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