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Semi Group Rings Which are Chinese Ring*

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In this paper we obtain conditions under which a semigroup ring is a Chinese ring. Further we define what are called weakly Chinese rings and study them. The authors in [1] called a commutative ring R to be a Chinese ring if, given elements $a, b \in R$ and ideal $I, J \subset R$ such that $a \equiv b(I+J)$ there exists an element $c \in R$ such that $c \equiv a(I)$ and $c \equiv b(J)$. For more properties about Chinese rings please refer [1].

Throughout this paper K denotes a field (or R denotes a commutative ring) and S denotes a commutative semigroup under multiplication. KS (or RS) is the semigroup ring of K (or R) over S. KS is the ring consisting of all finite formal sums $\sum_i \alpha_i s_i$ ($\alpha_i \in K, s_i \in S$), with the obvious definition of addition and with multiplication induced by the given multiplication in K and S according to the rule:

$$(\sum_i \alpha_i x_i)\,(\sum_j \beta_j y_j) = \sum_{i,j}(\alpha_i\beta_j)(x_i y_j) \qquad (\alpha_i, \beta_j \in K, x_i, y_j \in S).$$

For all $\alpha_i \in K$ and $s \in S$ we have $\alpha_i s = s\alpha_i$. If $1 \in K, 1 \cdot x = x$ for all $x \in S$.

Definition 1 *Let S be a commutative semigroup under multiplication and R a commutative ring. The semi group ring RS is called a Chinese ring if given elements $\alpha, \beta \in KS$ and ideals $A, B \subseteq KS$ such that $\alpha \equiv \beta(A+B)$ there eixsts an element $c \in KS$ such that $c \equiv \alpha(A)$ and $c \equiv \beta(B)$. [By putting the congruence $c \equiv \alpha(A)$ we mean $(A,c) = (A,\alpha)$ where ( ) denotes the ideal generation].*

Example 1 Let S be a finite semigroup having a subsemigroup T, where T is a group and K any field. Then KT is an Artinian ring, hence KT is a Chinese ring contained in KS.

Example 2 Let $Z_2 = (0,1)$ and $S = \{a, b, 0 | a^2 = a, b^2 = b, ab = ba = 0\}$ be a multiplicative semigroup. Z_2S is a Chinese ring. (Easy to verify).

Theorem 2 *Let S be a finite clifford semigroup and K any field. Then the semigroup ring KS is a union of Chinese rings.*

Proof S is a clifford semigroup, that is S is a union of groups. Consider the semigroup ring KS. Since S is a union of groups, KS is the union of group rings , say $KS = \cup KG_i$

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(where G_i is a group contained in S such that $S = \cup G_i$). Clearly every group ring is Artinian by [2], since each G_i is a finite group. But as an Artinian ring is a Chinese ring [1] we have KS to be the union of Chinese rings.

Theorem 3 *Let K be a prime field and S a finite commutative semigroup such that the set of all ideals under the order inclusion forms a finite chain. Then the semigroup ring KS is a Chinese ring.*

Proof Let S be a finite commutative semigroup. Let $\{I_i\}_{i=1}^{n}$ be the collection of all ideals of S such that $\{0\} = I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n = S$. Then $K \subseteq KI_1 \subseteq KI_2 \subseteq \cdots \subseteq KI_n = KS$ satisfies the chain conditions. Hence KS is Artinian and KS is a Chinese ring.

Definition *A semigroup ring KS is said to be a weakly Chinese ring if given any two ideals A, B in KS we have a pair of elements $\alpha, \beta \in KS \setminus \{A \cup B\}$ such that $(A, \alpha) = (B, \beta)$, where $\{A \cup B\}$ just denotes the set theoretic union of A and B and (A, α) denotes the ideal generated by A and α.*

Example Let $Z_2 = (0, 1)$ and $S = \{0, 1, a, b | a^2 = 1, b^2 = 0, \text{ and } ab = ba = b\}$ be the semigroup under multiplication. Then the semigroup ring $Z_2S = \{0, 1, a, b, 1 + a, 1 + b, a + b, 1 + a + b\}$ is a weakly Chinese ring. Now the only ideals of Z_2S are $\{0, b\}, \{0, a\}$ and $\{0, 1 + a + b\}$. Clearly these ideals satisfy the necessary conditions for the semigroup ring RS to be a weakly Chinese ring.

Theorem 5 *Let KS be the semigroup ring having only two proper ideals. Then KS is a weakly Chinese ring.*

Proof Let I and J be the only ideals of KS. So for every pair of elements α, β in $KS \setminus \{I \cup J\}$ we have $(I \cup \alpha) = (J \cup \beta) = KS$, since KS has only two proper ideals. Hence the theorem.

Theorem 6 *Let KS be a semigroup ring such that every ideal in it is maximal. Then KS is a weakly Chinese ring.*

Proof Let $\{I_j\}_{j \in A}$, A be an indexing set denoting the collection of all ideals. Every given ideal is maximal, so if $\alpha, \beta \in KS \setminus \{I_j \cup I_k\}$, where $I_j, I_k \in \{I_i\}_{i \in A}$, then clearly $(\alpha \cup I_j) = (\beta \cup I_k) = KS$. Hence the theorem follows.

References

[1] K.E. Aubert and I. Beck, *Chinese rings*, J. Pure and Appl. Algebra, Vol.24(1982), 221–226.

[2] D.S. Passman, *Infinite group rings*, Marcel Dekker, 1971.