Journal of Mathematical Research & Exposition
Vol.13, No.4, Nov. 1993

Besov Spaces of Ditzian-totik Type and Bernstein-Durrmeyer Operators*

Zhou Dingxuan (Advanced Institute of Math. Zhejiang Univ.)
Zhang Nansong (Dept.of Math., Zhejiang Univ.)

1. Introductions

For a function $f \in L_p[0,1]$, the n-th Bernstein-Durrmeyer operator is given by

$$D_n(f,x) = \sum_{k=0}^{n}(n+1)\int_0^1 f(t)P_{n,k}(t)dtP_{n,k}(x),$$

where $x \in [0,1]$ and $P_{n,k}(x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-k}$.

In this paper we use the following linear combinations of the operators $\{D_n\}$ to characterize Besov spaces of Ditzian-Totik type.

Definition 1.1 *The operator $D_{n,r}(n,r \in N)$ is defined by $D_{n,r}(f,x) = \sum_{i=0}^{2r-1} C_i(n)D_{n_i}(f,x)$, where $C_i(n)$ and n_i satisfy the following conditions:*

(1) $n_0 = n < n_1 < \cdots < n_{2r-1} < Kn$; *(2)* $\sum_{i=0}^{2r-1} |C_i(n)| < C$;

(3) $\sum_{i=0}^{2r-1} C_i(n) = 1$; *(4)* $\sum_{i=0}^{2r-1} C_i(n)n_i^{-\rho} = 0, \quad \rho = 1,2,\dots,2r-1,$

where K and C are constants independent of n.

For $1 \le p < \infty$, weighted Sobolev spaces are given by

$$W_{\varphi,p}^r = \{g \in L_p[0,1] : g^{(r-1)} \in \text{A.C.loc.}, \|g\|_{W_{\varphi,p}^r} < \infty\},$$

where $\varphi(x) = \sqrt{x(1-x)}$, $\|g\|_{W_{\varphi,p}^r} = \|g\|_p + \|\varphi^r g^{(r)}\|_p$. The Besov spaces of Ditzian-Totik type discussed in this paper are defined by Lizhong Peng and Ding-Xuan Zhou [6] as $B_{p,\frac{1}{2},\frac{1}{2}}^{s,q} = (L_p, W_{\varphi,p}^r)_{\frac{s}{r},q}$, where $0 < s < r,\ 1 \le q < \infty$.

The K-functional of Ditzian-Totik type is given by

$$K_{\varphi,r}(f,t^r)_p = \inf_{g \in W_{\varphi,p}^r} \{\|f-g\|_p + t^r\|g\|_{W_{\varphi,p}^r}\}.$$

*Received May 8, 1991. Supported by Zhejiang Provincial Science Foundation.

It was proved in [2], for $1 \le q < \infty$, and $n, r \in N, r > s > 0$, that

$$f \in B^{s,q}_{p,\frac{1}{2},\frac{1}{2}} \Longleftrightarrow \int_0^1 [t^{-\frac{s}{r}} K_{\varphi,r}(f,t)_p]^q \frac{dt}{t} < \infty.$$

We will show, for $1 \le p < \infty, 1 \le q < \infty$, and $n, r \in N, 2r > s > 0$, that

$$f \in B^{s,q}_{p,\frac{1}{2},\frac{1}{2}} \Longleftrightarrow \{\sum_{n=1}^{\infty} [n^{\frac{s}{2}} \|D_{n,r}(f) - f\|_p]^q \frac{1}{n}\}^{\frac{1}{q}} < \infty.$$

2. Lemmas for Bernstein-Durrmeyer Operators

In this section M is a constant independent of n or f. The following two lemmas were proved in [3].

Lemma 2.1 *For $f \in W^{2r}_{\varphi,p}$ and $1 \le p < \infty$, we have $\|\varphi^{2r} D_n^{(2r)}(f)\|_p \le \|\varphi^{2r} f^{(2r)}\|_p$.*

Lemma 2.2 *For $f \in L_p[0,1]$ and and $1 \le p < \infty$, we have $\|\varphi^{2r} D_n^{(2r)}(f)\|_p \le M n^r \|f\|_p$.*
By the above two lemmas and taking infinimum in $f \in W^{2r}_{\varphi,p}$, we can easily prove the following theorem.

Theorem 2.3 *For $1 \le p < \infty, f \in L_p[0,1], r \in N$, we have*

$$\|D_{n,r}(f)\|_{W^{2r}_{\varphi,p}} \le M n^r K_{\varphi,2r}(f, n^{-r})_p.$$

Lemma 2.4[3] *For $1 \le p < \infty, f \in L_p[0,1], r \in N$, we have*

$$\|D_{n,r}(f)\|_p \le M K_{\varphi,2r}(f, n^{-r})_p.$$

3. Main Results

In this section we will prove the following results.

Theorem 3.1 *For $1 \le p < \infty, 1 \le q < \infty$, and let $r \in N, 2r > s > 0$, we have*

$$\begin{aligned} f \in B^{s,q}_{p,\frac{1}{2},\frac{1}{2}} \quad &\Longleftrightarrow \quad \{\sum_{n=1}^{\infty} [n^{\frac{s}{2}} \|D_{n,r}(f) - f\|_p]^q \frac{1}{n}\}^{\frac{1}{q}} < \infty \\ &\Longleftrightarrow \quad n^{-\frac{1}{q}} n^{\frac{s}{2}} [D_{n,r}(f,x) - f(x)] \in l^q(L_p). \end{aligned} \tag{3.1}$$

Proof First we prove the direct result of (3.1). Using Lemma 2.4, we have

$$\begin{aligned} \sum_{n=1}^{\infty} [n^{\frac{s}{2}} \|D_{n,r}(f) - f\|_p]^q \frac{1}{n} \quad &\le \quad \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} [n^{\frac{s}{2}} M K_{\varphi,2r}(f, n^{-r})_p]^q n^{-1} \\ &\le \quad \sum_{k=0}^{\infty} [2^{(k+1)\frac{s}{2}} M K_{\varphi,2r}(f, 2^{-kr})_p]^q \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{r\ln 2}(M2^{r+\frac{s}{2}})^q \sum_{k=0}^{\infty} \int_{2^{-(k+1)r}}^{2^{-kr}} [t^{-\frac{s}{2r}} K_{\varphi,2r}(f,t)_p]^q \frac{dt}{t} \\
&\le \frac{1}{r\ln 2}(M2^{r+\frac{s}{2}})^q \int_0^1 [t^{-\frac{s}{2r}} K_{\varphi,2r}(f,t)_p]^q \frac{dt}{t} \\
&< \infty.
\end{aligned}
$$

We now prove the inverse result of (3.1). We take a constant $A \in N$, which will be determined later. For $k \in N$ we take $n_k \in N$, which satisfies the following conditions:

(1) $A^{\frac{k-1}{r}} \le n_k < A^{\frac{k}{r}}$; (2)$\|D_{n_k,r}(f) - f\|_p = \min\limits_{A^{\frac{k-1}{r}} \le l < A^{\frac{k}{r}}} \|D_{l,r}(f) - f\|_p$.

By Theorem 2.3 we derive by induction

$$
\begin{aligned}
A^{\frac{ks}{2r}} K_{\varphi,2r}(f, A^{-k})_p &\le A^{\frac{ks}{2r}} \|f - D_{n_k,r}(f)\|_p + MA^{(\frac{ks}{2r}-k)} n_k^r K_{\varphi,2r}(f, n_k^{-r})_p \\
&\le A^{\frac{ks}{2r}} \|f - D_{n_k,r}(f)\|_p + A^{k(\frac{s}{2r}-1)} [M n_k^r \|f - D_{n_{k-1},r}(f)\|_p \\
&\quad + M^2 n_{k-1}^r K_{\varphi,2r}(f, n_{k-1}^{-r})_p] \\
&\le \cdots \\
&\le A^{\frac{ks}{2r}} \|f - D_{n_k,r}(f)\|_p + A^{k(\frac{s}{2r}-1)} [\sum_{l=1}^{k-1} M^l n_{k-l+1}^r \|f - D_{n_{k-1},r}(f)\|_p \\
&\quad + M^k n_1^r K_{\varphi,2r}(f, n_1^{-r})_p] \\
&\le A^{1+\frac{s}{2r}} \sum_{l=0}^{k-1} (MA^{\frac{s}{2r}-1})^l [n_{k-l}^{\frac{s}{2}} \|f - D_{n_{k-l},r}(f)\|_p] \\
&\quad + A(MA^{\frac{s}{2r}-1})^k \|f\|_p.
\end{aligned}
$$

We now choose $A \in N$ such that $a := MA^{\frac{s}{2r}-1} < \frac{1}{2}$. For $1 < q < \infty$, we have

$$
\begin{aligned}
\int_0^{A^{-1}} [t^{-\frac{s}{2r}} K_{\varphi,2r}(f,t)_p]^q \frac{dt}{t} &\le A^{\frac{sq}{2r}} \ln A \sum_{k=0}^{\infty} [A^{\frac{ks}{2r}} K_{\varphi,2r}(f, A^{-k})_p]^q \\
&\le 2^q A^{\frac{sq}{2r}} (\ln A) A^{(1+\frac{s}{2r})q} \sum_{k=1}^{\infty} \{[\sum_{l=0}^{k-1} a^l n_{k-l}^{\frac{s}{2}} \\
&\quad \cdot \|f - D_{n_{k-l},r}(f)\|_p]^q + A^q (a^k \|f\|_p)^q\} \\
&\le A_1 A^q \frac{a}{1-a} \|f\|_p^q + 2^{q-1} A_1 \sum_{l=1}^{\infty} \sum_{k=l+1}^{\infty} a^{k-l} \\
&\quad \cdot [n_l^{\frac{s}{2}} \|f - D_{n_l,r}(f)\|_p]^q \\
&\le 2A_1 A^q \|f\|_p^q + 2^{q-1} A_1 \sum_{l=1}^{\infty} \sum_{k=l+1}^{\infty} a^{k-l} [n_l^{\frac{s}{2}} \|f - D_{n_l,r}(f)\|_p]^q \\
&\le 2A_1 A^q \|f\|_p^q + 2^q A_1 \sum_{l=1}^{\infty} [n_l^{\frac{s}{2}} \|f - D_{n_l,r}(f)\|_p]^q \\
&\le 2A_1 A^q \|f\|_p^q
\end{aligned}
$$

$$+ 2^q A_1 A^{\frac{s}{2r}} \sum_{l=1}^{\infty} \sum_{A^{\frac{l-1}{r}} \le m < A^{\frac{l}{r}}} [m^{\frac{s}{2}} \|f - D_{m,r}(f)\|_p]^q$$

$$< \quad \infty.$$

The proof for $p = 1$ is easy and we shall omit it.

4. Characterization of the Classical Besov Spaces $B^s_{p,q}$

In this section we use the linear combinations of the Gauss-Weierstrass operators to characterize the classical Besov spaces.

For $f \in L_p(-\infty, +\infty)$, the Gauss-Weierstrass operator is given by

$$W_n(f,x) = \sqrt{\frac{n}{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-n(t-x)^2}{2}} f(t)dt.$$

The linear combinations of operators $\{W_n\}$ are defined by $W_{n,r}(f,x) = \sum_{i=0}^{2r-1} C_i(n) W_{n_i}(f,x)$, where $C_i(n)$ and n_i satisfy the same conditions as in Definition 1.1.

For the operators $\{W_{n,r}\}$, it is easy to prove the following lemmas, where M denotes a constant independent of n or f.

Lemma 4.1 *For $1 \le p < \infty, f \in L_p^{2r}$, we have $\|W_{n,r}(f) - f\|_p \le Mn^{-r}\|f\|_{L_p^{2r}}$, where $L_p^{2r} = \{g \in L_p(-\infty,+\infty) : g^{(2r-1)} \in A.C.loc., g^{(2r)} \in L_p\}$; $\|g\|_{L_p^{2r}} = \|g\|_p + \|g^{(2r)}\|_p$.*

Lemma 4.2 *For $f \in L_p(-\infty,+\infty)$, we have $\|W_{n,r}(f)\|_{L_p^{2r}} \le Mn^r\|f\|_p$.*

Lemma 4.3 *For $f \in L_p^{2r}$, we have $\|W_{n,r}(f)\|_{L_p^{2r}} \le M\|f\|_{L_p^{2r}}$.*

From the above lemmas we can get the following theorem.

Theorem 4.4 *For $1 \le p < \infty, 1 \le q \le \infty$, and let $r \in N, 2r > s > 0$, we have*

$$\begin{aligned} f \in B^s_{p,q} &= (L_p, L_p^{2r})_{\frac{s}{2r},q} \\ &\Longleftrightarrow \{\sum_{n=1}^{\infty} [n^{\frac{s}{2}} \|W_{n,r}(f) - f\|_p]^q \frac{1}{n}\}^{\frac{1}{q}} < \infty \\ &\Longleftrightarrow n^{-\frac{1}{q}} n^{\frac{s}{2}} [W_{n,r}(f,x) - f(x)] \in l^q(L_p). \end{aligned}$$

The proof is similar to that of Theorem 3.1 and we omit it here.

5. On Baskakov-Durrmeyer Operators

In this section we use the linear combinations of Baskakov-Durrmeyer operators to characterize Besov Spaces of Ditzian-Totik type on $[0,\infty)$.

For $f \in L_p[0,\infty)$ the Baskakov-Durrmeyer operator is given by

$$V_n(f,x) = \sum_{k=0}^{\infty} P_{n,k}(x)(n-1) \int_0^1 P_{n,k}(t) f(t)dt,$$

where $x \in [0,\infty), P_{n,k}(x) = (-1)^k \frac{x^k}{k!}\varphi_n^{(k)}(x), \varphi_n(x) = (1+x)^{-n}$.

The linear combinations of the operators $\{V_n\}$ are defined by

$$V_{n,r}(f,x) = \sum_{i=0}^{2r-1} C_i(n) V_{n_i}(f,x),$$

where $C_i(n)$ and n_i satisfy the same conditions as in the Definition 1.1.

The following Lemmas were proved in [9].

Lemma 5.1 *For $1 \le p < \infty, f \in W_{\varphi,p}^{2r}[0,\infty), n, r \in N, n > 2r$, we have*

$$\|\varphi^{2r} V_n^{(2r)}(f)\|_p \le M \|\varphi^{2r} f^{(2r)}\|_p,$$

where $W_{\varphi,r}^{2r}[0,\infty) = \{g \in L_p[0,\infty) : g^{(2r-1)} \in A.C.loc. ?, \varphi^{2r} g^{(2r)} \in L_p\}$; $\varphi(x) = \sqrt{x(1+x)}$.

Lemma 5.2 *For $1 \le p < \infty, f \in L_p[0,\infty), n, r \in N, n > 2r$, we have*

$$\|\varphi^{2r} V_n^{(2r)}(f)\|_p \le M n^r \|f\|_p.$$

By the above lemmas it is easy to prove the following results.

Lemma 5.3 *For $1 \le p < \infty, f \in L_p[0,\infty), n, r \in N, n > 2r$, we have*

$$\|V_{n,r}(f)\|_{W_{\varphi,p}^{2r}} \le M n^r K_{\varphi,2r}(f, n^{-r})_p,$$

where $K_{\varphi,r}(f,t^r)_p = \inf\limits_{g \in W_{\varphi,p}^r} \{\|f-g\|_p + t^r \|g\|_{W_{\varphi,p}^r}\}$; $\|g\|_{W_{\varphi,p}^r} = \|g\|_p + \|\varphi^r g^{(r)}\|_p$.

Lemma 5.4[9] *For $1 \le p < \infty, f \in L_p[0,\infty), n, r \in N, n > 2r$, we have*

$$\|V_{n,r}(f) - f\| \le M K_{\varphi,2r}(f, n^{-r})_p.$$

Thus we can now obtain the following theorem.

Theorem 5.5 *For $1 \le p < \infty, 1 \le q < \infty, n, r \in N, n > 2r > s > 0$, we have*

$$\begin{aligned} f \in B_{p,\frac{1}{2},\frac{1}{2}}^{s,q}[0,\infty) &= (L_p[0,\infty), W_{\varphi,p}^{2r}[0,\infty))_{\frac{s}{2r},q} \\ &\iff \{\sum_{n=1}^{\infty} [n^{\frac{s}{2}} \|V_{n,r}(f) - f\|_p]^q \frac{1}{n}\}^{\frac{1}{q}} < \infty \\ &\iff n^{-\frac{1}{q}} n^{\frac{s}{2}} [V_{n,r}(f,x) - f(x)] \in l^q(L_p). \end{aligned}$$

The proof is similar to that of Theorem 3.1 and we omit it again.

Acknowledgement *The authors would like to thank Professor Guo Zhurui for his continuous guidance!*

References

[1] H.Berens & G. G.Lorentz, *Inverse theorems for Bernstein polynomials,* Indiana Univ. Math. J., **21**(1972), 693–708.

[2] Bergh & Lörfstrom, Interpolation Spaces, 1976.

[3] Z.Ditzian & K. G.Ivanov, *Bernstein-type operators and their derivatives,* J.Approx. Theory, **56**(1989), 72–90.

[4] Z.Ditzian & V.Totik, *Moduli of Smoothness,* Springer- Verlag, 1987.

[5] Guo Zhurui & Zhou Dingxuan, *Approximation theorems for modified Szász operators,* Accepted for Publication in Acta Sci. Math. (Szeged).

[6] Peng Lizhong & Zhou Dingxuan, *Besov Spaces of Ditzian- Totik type, to appear.*

[7] Zhou Dingxuan & Zhang Nansong, *Direct and inverse theorems for Jackson operators in Besov spaces, to appear.*

[8] Zhou Dingxuan, *L^p-Inverse theorems for Beta operators,* Accepted for Publication in J. Approx. Theory.

[9] M.Heilman, *Direct and converse results for operators of Baskakov-Durrmeyer type,* Approx. Theory and its Appl., **5:1**(1989), 105– 127.

[10] Zhou Dingxuan, *On a conjecture of Z.Ditzian,* Accepted for publication in J. Approx. Theory.

Ditzian—Totik 型的 Besov 空间和 Bernstein—Durrmeyer 算子

周定轩　　　　张南松

（浙江大学数学系，杭州 310027）（浙江农业大学，杭州）

摘　要

对任一函数 $f\in Lp[0,1]$，n 阶 Bernstein—Durrmeyer 算子为

$$D_n(f,x)=\sum_{k=0}^{n}(n+1)\int_0^1 f(t)P_{n,k}(t)dtP_{n,t}(t),$$

其中 $x\in[0,1]$，$P_{n,k}(x)=\binom{n}{k}x^k-(1-x)^{n-k}$. 本文利用算子列 $\{D_n\}$ 的线性组合 $D_{n,r}(f,x)$ 刻画了 Ditzian—Totik 型的 Besov 空间，即证明了

$$f\in B^{s,q}_{p,\frac{1}{2},\frac{1}{2}}\Leftrightarrow\{\sum_{n=1}^{\infty}[n^{\frac{s}{2}}\|D_{n,r}(f)-f\|_p]^q\frac{1}{n}\}^{\frac{1}{q}}<\infty\Leftrightarrow\int_0^1[t^{-\frac{s}{r}}K_{\varphi,r}(f,t)_p]^q\frac{dt}{t}<\infty.$$