Journal of Mathematical Research & Exposition
Vol.13, No.4, Nov. 1993

On Extreme Stability of Nonlinear Systems and Its Applications *

Zhang Yusen
(Scientia Sinica, Academia Sinica, Beijing 100707)

Abstract By generalizing and developing C. V.Pao's inner product method in studying the Liapunov stability, this paper deals with the extreme stability, Liapunov stibility, existence and uniqueness of the stationary position of nonlinear systems. The main results are applied to determine the stable oscillations of nonlinear periodic systems.

1. Introductions

Consider the nonlinear nonautonomous differential system

$$\frac{dx}{dt} = A(t)x + f(t,x), \tag{1.1}$$

where $x \in R^n, t \in I = [t_0, +\infty), A(t) = (a_{ij}(t))_{n\times n}$ is a real continuous matrix on $I, f(t,x) : I \times R^n \to R^n$ is continuous and ensures the uniqueness of Cauchy solutions of (1.1).

Liapunov stability of the solutions of (1.1) has been studied enormously mainly with V function method in which the meritis that the solutions of (1.1) are not necessary while the shortness is that there is no general method and rule of constructing V functions. Besides, to study the stability of some solution $\tilde{x}(t)$, one always assumes that $x(t)$ is given and then (1.1) can be reduced by translations $y(t) = x(t) - \tilde{x}(t)$ into

$$\frac{dy}{dt} = A(t)y + f(t, y + \tilde{x}(t)) - f(t, \tilde{x}(t)) \triangleq G(t,y), \tag{1.2}$$

and the problem becomes to study the stability of the trivial solution of (1.2). Hence, the criterion based on (1.2) is very differcult to be verified.

On the other hand, the extreme stability introduced by LaSalle, Lefschitz[2] and Yoshizawa does not depend on certain solutions and is a kind of stronger stability.

C. V.Pao[1,4,5] introduced the inner-production method which allows the norm induced from the inner production of a Hilbert space to replace V-function while still preserving V-functions advantages. Also, one is able to avoid the difficulty of constructing V-function.

*Received Nov. 12, 1991.

By revising and developing inner-product method, and applying it to a kind of more general dissipative differential operators, this paper obtains a series of results regarding extreme stability and extremely asymtotical stability of (1.1). These results take all the results of [1] as special cases. We introduce the concept of extreme unstability and its criterion. Also, the existence, uniqueness and Liapunov stabibity of the equilibrium of (1.1) are studied. As an application, a concrete criterion for the existence of stable oscillation of nonlinear periodic system is given.

2. Definitions and Lemmas

Let $x = \mathrm{col}(x_1, \cdots, x_n) \in R^n, y = \mathrm{col}(y_1, \cdots, y_n) \in R^n$. The inner-product of x, y is

$$\langle x, y\rangle \triangleq x^T y \triangleq \sum_{i=1}^{n} x_i y_i, \tag{2.1}$$

which induces a norm

$$\|x\| = \langle x, x\rangle^{\frac{1}{2}}.$$

Lemma 2.1 *Let V be an $n \times n$ real symmeteric positive definite matrix, and an inner product be defined as*

$$\langle x, y\rangle_V \triangleq \langle x, Vy\rangle, \quad x, y \in R^n, \tag{2.2}$$

$$\|x\|_V^2 \triangleq \langle x, Vx\rangle.$$

Then there exist constants $\delta > 0$ and $\gamma > 0$, such that

$$\delta\|x\|^2 \leq x\|_V^2 \leq \|\gamma\|x\|^2, \tag{2.3}$$

i.e., inner products (2.1) and (2.2) are equivalent.

Proof Omitted (see [1], Lemma 2.1).

Definition 2.1 *Operator A (linear or nonlinear) is said to be generalized dissipative regarding to inner product (2.1), if there exists $C = diag(c_1, \cdots, c_n) > 0$ (i.e., $c_i > 0$, $i = 1, \ldots, n$), such that*

$$\langle A(x-y), C(x-y)\rangle \leq 0, \quad \forall x, y \in R^n.$$

Operator A (linear or nonlinear) is said to be generalized strongly dissipative regarding to inner product (2.1), if there exists $C = diag(c_1, \cdots, c_n) > 0$ and a constant $\beta > 0$ such that

$$\langle A(x-y), C(x-y)\rangle \leq -\beta\langle x-y, x-y\rangle, \quad \forall x, y \in R^n.$$

Lemma 2.2 *A necessary and sufficient condition for the matrix $A = (a_{ij})_{n\times n}$ to be a generalized dissipative operator (generalized strongly dissipative operator) regarding to inner product (2.1) is that there exists $C = diag(c_1, \cdots, c_n) > 0$ such that $A^TC + CA$ is semi-negative definite (negative definite).*

Proof Since

$$\begin{aligned}\langle A(x-y), C(x-y)\rangle &= (A(x-y))^T C(x-y) = (x-y)^T A^T C(x-y)\\ &= \frac{1}{2}(x-y)^T(A^TC + CA)(x-y)\\ &\le \begin{cases} 0, \text{ when } A^TC+CA \text{ is semi-negative definite;}\\ -\beta\langle(x-y),(x-y)\rangle, \text{ when } A^TC+CA \text{ is negative definite.}\end{cases}\end{aligned}$$

The lemma is proved.

In the following, $x(t,t_0,x_0), y(t,t_0,y_0)$ denote the solutions of (1.1), passing through initial point $(t_0,x_0),(t_0,y_0)$ respectively.

Definition 2.2 *Solutions of (1.1) are extremely stable, if* $\forall\varepsilon>0, \exists\delta(\varepsilon,t_0)>0, \forall x_0, y_0 \in R^n$, *we have*

$$\|x(t,t_0,x_0)-y(t,t_0,y_0)\|<\varepsilon, \quad \textit{for } t\ge t_0,\ \|x_0-y_0\|<\delta.$$

Definition 2.3 *Solutions of (1.1) are extremely attractive, if* $\forall x_0, y_0\in R^n$, *we have*

$$\|x(t,t_0,x_0)-y(t,t_0,y_0)\|\to 0 \quad (t\to+\infty);$$

solutions of (1.1) are extremely asymptotically stable, if solutions of (1.1) are extremely stable and extremely attractive; solutions of (1.1) are extremely exponentially stable, if there are constants $M>0, \alpha>0$ *such that*

$$\|x(t,t_0,x_0)-y(t,t_0,y_0)\|\le Me^{-\alpha(t-t_0)}.$$

Definition 2.4 *Solutions of (1.1) are strongly (weakly) extremely unstable, if* $\exists\varepsilon_0>0, \forall\delta>0$ *for* $\forall x_0\ne y_0\in R^n (\exists x_0\ne y_0\in R^n)$ *and* $\|x_0-y_0\|<\delta$, $\exists t_1>t_0$ *such that*

$$\|x(t_1,t_0,x_0)-y(t_1,t_0,y_0)\|\ge\varepsilon_0.$$

Lemme 2.3 *Suppose that (1.1) satisfies the conditions ensuring the existence and uniqueness of the solutions, there exists an* $w=$*const.*>0 *with* $A(t+w)\equiv A(t), f(t+w,x)\equiv f(t,x)$ *for* $\forall t\in I, \forall x\in R^n$, *solutions of (1.1) are extremely attractive, and that (1.1) has a bounded solutions. Then (1.1) has a unique* w*-periodic solution, and all solutions tend to this periodic solution as* $t\to+\infty$.

Proof Omitted (see [2]).

In this case system (1.1) is called a stable oscillations system.

3. Extreme Stability of Non-autonomous Systems

In the following we discuss the extreme stability of solutions of (1.1).

Theorem 3.1 *Let matrix $A(t) \in C[t_0, +\infty)$ and $f(t,x) : [t_0, +\infty) \times R^n \to R^n$ be continuous. If there exists a diagonal matrix $C = diag(c_1, \cdots, c_n) > 0$, such that $\forall x, y \in R^n, t \in [t_0, +\infty)$ imply*

$$\langle f(t,x) - f(t,y),\ C(x-y)\rangle \leq K(t)\|x-y\|^2 (K(t) \in [t_0, +\infty)).$$

Then, for any solutions $x(t)$ and $y(t)$ of (1.1), we have the evaluation

$$\|x(t) - y(t)\| \leq \bar{C}/\underline{C}\|x(t_0) - y(t_0)\| \exp\int_{t_0}^{t} (\gamma(s) + K(s))/\underline{C} ds,$$

where $\gamma(t)$ is the largest characteristic value of $\frac{1}{2}(A^T(t)C + CA(t))$, $\bar{C} \triangleq \max\limits_{1\leq i<n} c_i$, $\underline{C} \triangleq \min\limits_{1\leq i\leq n} c_i$. Therefore,

1) $\int_{t_0}^{+\infty}(\gamma(s) + K(s))ds < +\infty (t \geq t_0)$ implies that the solution of (1.1) are extremely stable;

2) $\int_{t_0}^{+\infty}(\gamma(s) + K(s))ds = -\infty$ implies that the solution of (1.1) are extremely asymtotically stable;

3) $\int_{t_0}^{+\infty}(\gamma(s) + K(s))ds \leq -\alpha(t - t_0)$ $(t \geq t_0)$ implies that the solutions of (1.1) are extremely exponentially stable.

Proof Let $x(t), y(t)$ be solutions of (1.1), $z(t) = x(t) - y(t), \tilde{f}(t,z) \triangleq f(t,x) - f(t,y)$. Then $z(t)$ is a solution of

$$\frac{dz}{dt} = A(t)z + \tilde{f}(t,z). \tag{3.1}$$

Since C is symmetric positive definite, $\langle Cz, z\rangle$ is positive difinite. It is easy to prove that

$$\frac{d}{dt}\langle Cz, z\rangle \Big|_{(3.1)} \leq \frac{2(\gamma(t) + K(t))}{\underline{C}}\langle Cz, z\rangle \tag{3.2}$$

Integrating (3.2) from t_0 to t, we have

$$\langle Cz(t), z(t)\rangle \leq \langle Cz(t_0), z(t_0)\rangle \exp\int_{t_0}^{t} \frac{2(\gamma(s) + K(s))}{\underline{C}} ds. \tag{3.3}$$

It follows that

$$\underline{C}\|z(t)\|^2 \leq \langle Cz(t), z(t)\rangle \leq \bar{C}\|z(t_0)\| \exp\int_{t_0}^{t} \frac{2(\gamma(s) + K(s))}{\underline{C}} ds, \tag{3.4}$$

and

$$\|x(t) - y(t)\| \leq (\frac{\bar{C}}{\underline{C}})^{\frac{1}{2}}\|x(t_0) - y(t_0)\| \exp\int_{t_0}^{t} \frac{(\gamma(s) + K(s))}{\underline{C}} ds. \tag{3.5}$$

This proves all the conclusions of Theorem 3.1.

Remark If $C \equiv E$ (unity matrix) we obtain Theorems 3.1, 3.2, 3.3 of [1]. Obviously, our conditions are weaker than these theorems.

Theroem 3.2 *Let $A(t) \in C[t_0, +\infty)$, $f(t,x) : [t_0, +\infty) \times R^n \to R^n$ be continuous. Soppose that there exists a function $K(t) \in C[t_0, +\infty)$ such that*

$$\|f(t,x) - f(t,y)\| \le K(t)\|x - y\|,$$

and $C = diag(c_1, \cdots, c_n) > 0$. Then

1) $\int_{t_0}^{+\infty}[\gamma(s) + \bar{C}K(s)]ds < +\infty, t \ge t_0$, implies that solutions of (1.1) are extremely stable;

2) $\int_{t_0}^{+\infty}[\gamma(s) + \bar{C}K(s)]ds = -\infty, t \ge t_0$, implies that solutions of (1.1) are extremely asymptotically stable;

3) $\int_{t_0}^{t}[\gamma(s) + \bar{C}K(s)]ds \le -\alpha(t - t_0), t \ge t_0$, implies that solutions of (1.1) are extremely exponentially stable, where $\gamma(t)$ is the largest characteristic value of $\frac{1}{2}(CA + A^TC)$.

The proof is obvious.

Theorem 3.3 *Let $A(t) \in C[t_0, +\infty)$, $f(t,x) : [t_0, +\infty) \times R^n \to R^n$ be continuous. Soppose that there exists a function $K(t) \in C[t_0, +\infty)$ and $C = diag(c_1, \cdots, c_n) > 0$ such that*

$$\langle f(t,x) - f(t,y), C(x - y)\rangle \ge K(t)\|x - y\|$$

and

$$\int_{t_0}^{t}(\beta(s) + K(s))ds \text{ is unbounded,}$$

where $\beta(t)$ is the smallest characteristic value of $\frac{1}{2}(CA + A^TC)$. That solutions of (1.1) are strongly extremely unstable.

Proof Let $z \triangleq x - y$, $f(t,z) \triangleq f(t,x) - f(t,y)$. Because $\langle Cz, z\rangle$ is positive definite, we have

$$\frac{d}{dt}\langle Cz, z\rangle \Big|_{(3.1)} \ge 2[\beta(t) + K(t)]\frac{Cz, z}{\bar{C}}.$$

It follows that

$$\langle Cz, z\rangle \ge \langle Cz(t), z(t_0)\rangle \exp\int_{t_0}^{t}\frac{2(\beta(s) + K(s))}{\bar{C}}ds$$

and

$$\bar{C}\|z\|^2 \ge \langle Cz, z\rangle \ge \langle Cz(t_0), z(t_0)\rangle \exp\int_{t_0}^{t}\frac{2(\beta(s) + K(s))}{\bar{C}}ds. \tag{3.3}$$

(3.3) shows that Theorem 3.3 is ture.

4. Applications in Stable Oscillations

Consider the following nonlinear periodic system,

$$\frac{dx}{dt} = A(t)x + f(t,x), \tag{4.1}$$

where $A(t+\omega) = A(t), f(t+\omega, x) \equiv f(t, x), \omega = \text{const.} > 0,\ A(t) \in C[t_0, +\infty), f(t, x)$: $(t_0, +\infty) \times R^n \to R^n$ is continuous and ensures the uniqueness of solutions.

Theorem 4.1 *Suppose that (4.1) has bounded solutions, and that there exists a matrix*

$$C = diag(c_1, \cdots, c_n) > 0$$

such that $\forall x, y \in R^n, t \in [t_0, +\infty)$ *imply*

$$\langle f(t,x) - f(t,y), C(x-y)\rangle \le K(t)\|x-y\|^2, \quad K(t) \in C[t_0, +\infty)$$

and

$$\int_{t_0}^{+\infty} (\gamma(s) + K(s))ds = -\infty,$$

where $\gamma(s)$ *is the largest characteristic value of* $\frac{1}{2}(CA + A^T)$*. Then system (4.1) is a stable oscillatory system, i.e., (4.1) has a asymptotically stable* ω*-periodic solution.*

Theorem 4.2 *If (4.1) has bounded solutions, and there exists a function* $K(t) \in C[t_0, +\infty)$ *and* $C = diag(c_1, \cdots, c_n) > 0$ *such that*

$$\|f(t,x) - f(t,y)\| \le K(t)\|x - y\|$$

and

$$\int_{t_0}^{+\infty} (\gamma(s) + \bar{C}K(s))ds = -\infty,$$

where $\gamma(s)$ *is the largest characteristic value, then system (4.1) is a stable oscillatory system.*

When the conditions of Theorem 4.1 and Theorem 4.2 are satisfied, Theorem 3.1 and Theorem 3.2 ensure that the solutions of (4.1) are extremely asymptotically stable. By Lemma 2.3, system (4.1) is a stable oscillatory system.

Corollary *Consider n-dimensional linear homogeneous systems as follows:*

$$\frac{dx}{dt} = A(t)x + f(t), \tag{4.2}$$

where $A(t), f(t)$ *are continuous on* $[t_0, +\infty)$*. Assume* $\exists \omega > 0$ *such that*

$$A(t+\omega) \equiv A(t), f(t+\omega) \equiv f(t)$$

and there exists $C = diag(c_1, \cdots, c_n) > 0$ *with*

$$\int_{t_0}^{\infty} \gamma(s)ds = -\infty,$$

where $\gamma(s)$ *is the largest characteristic value of* $\frac{1}{2}(CA + A^T C)$*. By the well-known methods in ordinary differential theory, we can easily prove that (4.2) has bounded solutions (see the example below). Hence (4.2) is a stable oscillatory system.*

5. Determination of Uniqueness of Stationary State

In the following, we disscuss the uniqueness of stationary state-constant solution of (1.1).

Theorem 5.1 *Suppose that n-order matrix $A(t) \in C[t_0, +\infty)$, $f(t,x) : [t_0, +\infty) \times R^n \to R^n$ is continuous, $f(t,0) \equiv 0$, and that there exists a function $K(t) \in C[t_0, +\infty)$ and $C \doteq diag(c_1, \cdots, c_n) > 0$ such that*

$$\langle f(t,x), Cx \rangle \le K(t) \|x\|^2, \;\; x \in R^n, \;\; t \in [t_0, +\infty), \;\; \textit{and} \; \exists t_1 \in [t_0, +\infty)$$

such that

$$\gamma(t_1) + K(t_1) < 0$$

where $\gamma(t)$ is the largest characteristic value of $\frac{1}{2}(CA + A^T C)$. Then the trivial solution $x = 0$ of (1.1) is the unique stationary position.

Proof The proof of Theorem 3.1 shows that

$$\langle Cx(t), x(t) \rangle \le \langle Cx(t_0), x(t_0) \rangle \exp \int_{t_0}^{t} \frac{2(\gamma(s) + K(s))}{\underline{C}} ds.$$

Since $\gamma(t_1) + K(t_1) < 0$ and $\gamma(t), K(t)$ are continuous at t_1, there exists an interval (t', t'') such that $t_1 \in (t', t'') \subset (t_0, +\infty)$ and

$$\gamma(t) + K(t) < 0, \quad t \in (t', t'').$$

Let $\tilde{x}(t) \equiv \tilde{C} \neq 0$ be another stationary state, then

$$\tilde{x}(t') \neq \tilde{x}(t'').$$

Hence we have,

$$\begin{aligned} \langle C\tilde{x}(t''), \tilde{x}(t'') \rangle &\le \langle C\tilde{x}(t'), \tilde{x}(t') \rangle \exp \int_{t'}^{t''} \frac{2(\gamma(s) + K(s))}{\underline{C}} ds. \\ &< \langle C\tilde{x}(t'), \tilde{x}(t') \rangle . (\text{ since } \gamma(t) + K(t) < 0 \text{ for } t \in (t', t'')) \end{aligned}$$

but on the other hand $\langle C\tilde{x}(t''), \tilde{x}(t'') \rangle = \langle C\tilde{x}(t'), \tilde{x}(t') \rangle$. This contradiction shows $x(t) \equiv 0$. The theorem is proved.

Theorem 5.2 *Suppose*

1) n-order matrix $A(t) \in C[t_0, +\infty)$ and $f(t,x) : [t_0, +\infty) \times R^n \to R^n$ are continuous.

2) $f(t,0) \equiv 0, \exists K(t) \in C[t_0, +\infty)$ and $\langle f(t,x), Cx \rangle \ge K(t) \|x\|^2, x \in R^n, t \in [t_0, +\infty)$, $\exists t \in (t, +\infty)$, such that

$$\beta(t_1) + K(t_1) > 0,$$

where $\beta(t)$ is the smallest characteristic value of $\frac{1}{2}(CA + A^T C)$. Then the trivil solution is the unique stationary point of system (1.1).

Proof The proof is similar to that of Theorem 5.1.

In the following, we consider a kind of autonomous, nonlinear system:

$$\frac{dx}{dt} = Ax + f(x),$$

where $x \in R^n, A(a_{ij})_{n\times n}$, $f(x) : R^n \to R^n$ is continuous, $f(0) = 0$.

Theorem 5.3 *If there exists a sysmetric positive definite $n \times n$ matric V such that*

$$\langle f(x) - f(y), V(x-y)\rangle \geq K\|x-y\|^2, \ \beta + K > 0,$$

where K is a negative const., β is the smllest characteristic value of $\frac{1}{2}(VA + A^TV)$, then (5.1) has a unique stationary position $x = 0$.

The proof is omitted.

The following result can be established by using theorem 5.3.

Theorem 5.4 *If there exists a symmetric positive definite $n \times n$ matrix V such that*

$$\langle f(x) - f(y), V(x-y)\rangle \leq \gamma\|x-y\|^2,$$

γ is a negative constant, and $\alpha + \gamma < 0$, where α is the largest characteristic value of $\frac{1}{2}(VA + A^TV)$, then (5.1) has a unique stationary position $x = 0$.

Remark Results in this section are appliable to studying nonlinear algebric systems and transcendent systems.

6. Liapunov Stability of Trivil Solution

In the following, we discuss the stability in Liapunov's meaning of the trivil solution of (1.1).

Assume

$$f(t,0) \equiv 0, A(t) \in C[t_0, +\infty), f(t,x) : [t_0, +\infty) \times R^n \to R^n$$

is continuous.

Theorem 6.1 *Suppose that $\lim\limits_{t\to+\infty} A(t) \triangleq \tilde{A}$ exists, and there exists a sysmmetric positive difinite matrix V such that*

$$\langle f(t,x), Vx\rangle \leq K(t)\|x\|^2.$$

Let μ be the largest characteristic volue of $\frac{1}{2}(VA + A^TV)$. If there exists a sufficiently small constant $\varepsilon > 0$, then the following assumptions are the respective sufficient conditions for the (i) stability, (ii) globally asymptotic stability, (iii) globally exponential stability of the trivil solution of (1.1):

(i) $\int_{t_0}^{t}(\mu + \varepsilon + K(s))ds$, *is bounded* $(t \geq t_0)$;

(ii) $\int_{t_0}^{+\infty}(\mu + \varepsilon + K(s))ds = -\infty$;

(iii) $\int_{t_0}^{t}(\mu + \varepsilon + K(s))ds \leq -2(t - t_0), \alpha > 0, t \geq t_0$.

Proof $\lim_{t\to\infty} A(t) = \tilde{A}$ implies that for $\varepsilon > 0, \exists T > t_0$ such that

$$X^T[(A(t)-\tilde{A})^T V + V(A(t)-\tilde{A}]x \le 2\varepsilon\|x\|^2, t \ge T.$$

Conditions (i)$\longrightarrow$(iii) are respectively corresponding to

$$\int_{T_1}^{t}(\mu+\varepsilon+K(s))ds \text{ is bounded, } t \ge t_0;$$
$$\int_{T_1}^{+\infty}(\mu+\varepsilon+K(s))ds = -\infty;$$
$$\int_{T_1}^{t}(\mu+\varepsilon+K(s))ds = -\alpha(t-T_1)+\mu,\ (t \ge T_1, \mu \text{ is a constant}).$$

Besides, we can only deal with the stability of the trivial solution of (1.1) for $t \ge T_1$. Consider the inner product $\langle x, Vx\rangle$,

$$\begin{aligned} \frac{d}{dt}\langle x, Vx\rangle\Big|_{(1.1)} &= X^T(A^T(t)V + V(A(t))x + 2\langle f(t,x), Vx(t)\rangle \\ &\le \frac{2(\mu+\varepsilon+K(t))}{\gamma}(x, Vx),\quad t \ge T_1, \gamma \text{ is a constant.} \end{aligned}$$

We have

$$\langle x(t), Vx(t)\rangle \le \langle x(T), Vx(T_1)\rangle \exp\int_T^t \frac{2(\mu+\varepsilon+K(s))}{\gamma}ds,\quad t \ge T_1.$$

The last inequality leads to the desired conclusion.

Theorem 6.2 *If $\|f(t,x)\| \le e(t)\|x\|$, $\lim_{t\to+\infty} A(t) = \tilde{A}$ and there exists an $\varepsilon > 0$ and symmetric positive definite matrix V. Then the following conditions ensure respectively the stability, asymptotic stability and exponential stability of the trivial solution of (1.1), where μ is the largest characteristic value of $\frac{1}{2}(V\tilde{A} + \tilde{A}^T V)$, and $\gamma > 0$) satisfies (2.3):*

$$\int_{t_0}^{t}(u+\varepsilon+\gamma^2 e(s))ds \textit{ is bounded for } t \ge t_0,$$
$$\int_{t_0}^{+\infty}(u+\varepsilon+\gamma^2 e(s))ds = -\infty,$$
$$\int_{t_0}^{t}(u+\varepsilon+\gamma^2 e(s))ds \le -\alpha(t-t_0),\quad \alpha > 0.$$

Proof Since

$$\begin{aligned} \langle f(t,x), Vx\rangle &\le \langle f(t,x), x\rangle_V \le \|f(t,x)\|_V\|x\| \le \gamma\|x\|\gamma\|f(t,x)\| \\ &\le \gamma^2 e(t)\|x\|^2 \triangleq K(t)\|x\|^2, \end{aligned}$$

all the assumptions of Theorem 6.1 are satisfied.

References

[1] C. V.Pao, *On stability of non-linear differential systems*, Int. Non-Linear Mechanics, **8**(1973), 210–238.

[2] J.LaSalle & S.Lefschetz, *Stability by Liapunov's Direct Method with Applications*, Academic Press, New York, 1961.

[3] T.Yoshizawa, *Stability Theory by Liapunov's Second Method,* The Mathematical Society of Japan Tokyo, 1966.

[4] C. V.Pao, *Semi-groups and asymptotic stability of nonlinear differential equations*, SIAM J. Math. Analysis, **3**(1972), 371–379.

[5] C. V.Pao & W. G.Vogt, *On the stability of non-linear operator differential equation and applications,* Archive Rational Mech. Analysis, **35**(1969), 30–46.

[6] Qin Yuanxun, Wang Muqiu & Wang Lian, *Theory and Applications of Stability of Motion*, Academic Press(PRC), 1981.

关于非线性系统的非常稳定性及其应用

张 玉 森

(中国科学编辑部,北京 100707)

摘 要

本文将 C. V. Pao[1] 研究 Liapunov 稳定性的内积法改进、推广和发展,研究了非线性系统的非常稳定性、平衡位置的存在唯一性和 Liapunov 稳定性.将主要结果直接应用到非线性周期系统的稳态振荡的判定.