Journal of Mathematical Research & Exposition
Vol.13, No.4, Nov. 1993

Comment on Lieberman's Book Review*

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Prof. G. M.Lieberman published his book review in the 1993 January issue of Bull. of AMS on my monograph "Nonlinear Partial Differential Equations of Second Order" (Trans. Math. Monograph, Vol. 95 AMS). I would like to express my point of view on the aspects of scholarship, content of my monograph, style, reference literature, and the content of the book review. Hopefully, this would clarify matters and ensure a correct understanding of the facts.

I. The scholarship

Prof. Lieberman commented that a Hölder estimate for the solution in Chapter V of my monograph is a simple corollary of a work of DiBenedetto and Friedman [1]. The following equation occurs in the paper of DiBenedetto and Friedman,

$$u_t = \frac{\partial}{\partial x_i}[a_{ij}(x,t,u,u_x)v(u)u_{x_j}] + f(x,t,u,u_x), \quad \lambda I \leq (a_{ij}) \leq \Lambda I,$$

where $v(u)u_{x_j} = (u^m)_{x_j}$, m, λ and Λ are positive constants, $m > 1$. If $|f| \leq C_1|u^{m-1}u_x| + C_2$, then the solution u is Hölder continuous. With the equation unchanged and under the condition of $|f| \leq C_1|u^{(m-1)/2}u_x| + C_2$ in the case of $v(r) = mr^{m-1}$, Y. Z. Chen [2] proved that the solution is Hölder continuous. With weaker constraint of f, it is more difficult to get a Hölder estimate for the solution. In my monograph, the same conclusion holds if the restriction on f (in the case of $v(r) = mr^{m-1}$ also) is reduced to $|f| \leq C_1|u_x| + C_2$. It seems that my work on a priori estimate for the solution of a degenerated quasilinear parabolic equation contains the most advanced result. Prof. Lieberman mistakenly concluded that my above result is only a simple corollary of the result of DiBenedetto and Friedman in 1985 [1]. I am afraid that readers will be misled by him.

II. Content

The book is a monograph which summarizes the results of research completed by me and some of my students. Our results are presented through Chapter 2 to Chapter 10. The classical results of O. A.Ladyzenskaya and N. N. Ural'tseva are introduced in the first chapter. I consider that such a layout is reasonable. Prof. Lieberman also mentioned that parts of my monograph do not contain the latest result. However, for a book of this size, it is quite impractical to be uniformly up-to-date and at the same time keep the

*Received May 18, 1993.

number of pages from increasing. I confess that there exist some faults in the monograph. For instance, an unsolved problem mentioned in the first chapter was actually solved by DiBenedetto in 1986. However, the first chapter of the Chinese edition of my book was finished in 1985. At that time, the problem was unsolved indeed. But I failed to mention that the problem was solved by DiBenedetto.

III. The Style

Prof. Lieberman said that "Chapters I-IV are separate stories, Chapter V and VI form another little story, and then Chapter VII-X tell a final story, disjoint from the previous ones". In fact, there exists a main line throughout my monograph, that is, a priori estimates under different conditions. Prof. Lieberman gave an example to show that in a certain proof of Chapter I, the assumption may be weakend a little by using a result in Chapter IX. Then he used it as a typical example for the loose style of my monograph. I think this is unreasonable.

IV. Reference Literature

Prof. Lieberman said that "... even more puzzling is the absence of Krylov's book from the references". I am puzzled why Prof. Lieberman did not investigate carefully the relationship between the contents of Krylov's book and the contents of the later chapters of my book. He simply skimmed over the references in the monograph but ignored four of Krylov's papers which appeared in my references. In fact, the predominant contents in Krylov's book that I need in my monograph are contained in those four papers that I quoted. Thus it is not necessary to refer to Krylov's book.

V. The Content of the Book Review

We believe that the book reviewer should place his emphasis on the evaluation of the contents of the book. He should not lay stress on the work done by the reviewer himself [3] and not criticize matters of the secondary importance.

I select a few example to show some important results in my monograph.

1. Subsonic flow around an obstacle. When $n = 3$, the best result in the past is that the solution exists uniquely provided that the mach number is less than 0.7. In my book, I show that the solution exists uniquely provided that the mach number is less than 1 when $n \geq 3$. This result is optimal.

2. Hölder estimate for the solution of the degenerate parabolic equations of porous medium. This result is also optimal.

3. Krylov's estimate of fully nonlinear parabolic equations and proof of existence of the solution. My book contains many simplifications and generalizations of the theory of Krylov. Chapter VIII in my book discusses the density theorem, which is originally due to Krylov. I gave an alternative formulation which facilitates easier applications, e.g. in a paper that I delivered at an international conference on P.D.E held in Hangzhou.

4. DiBenedetto's result on 1986 on Hölder estimate for the equation

$$u_t = D_{x_i} a_i(x,t,u,p) + b(x,t,u,p),$$

$$\begin{aligned}\lambda(1+|p|^{m-2})I &\leq (\frac{\partial a_i}{\partial p_j}) \leq \Lambda(1+|p|^{m-2})I,\\ |b| &\leq C(1+|p|^m) \ \ (m>2).\end{aligned}$$

What I tackle in my book are more difficult Hölder estimates for the solution of quasi-linear and fully non-linear equations in non-divergence form

$$\begin{aligned}u_t &= F(x,t,u,p,r),\\ \lambda(1+|p|^{m-2})I &\leq F_{r_{ij}} \leq \Lambda(1+|p|^{m-2})I,\\ |F(x,t,u,p,0)| &\leq C(1+|p|^m) \ \ (m>2).\end{aligned}$$

Even for the quasi-linear case, I have partly solved an open problem proposed in the will-known monograph "The Linear and Quasi-linear Parabolic Equation" by O.A.Ladyzenskaya, V.A.Solonnikov and N. N. Ural'tseva. For the fully nonlinear equations, I have proved the existence and uniqueness of the solution with natural structure conditions of the second kind. My difficult a priori estimates have not been done by others.

Because my monograph contains the above material, it was recommended by the acknowledged authority in the field of P.D.E.. Professor Louis Nirenberg, to the American Mathematical Society. Later the American Mathematical Society's Committee on Translation from Chinese invited a number of experts to examine my book. The members of the Committee decided to publish it in the Translations of Mathematical Monographs of the AMS and selected an outstanding translator to accomplish it. Quite a number of internationally prominent scholars reviewed my monograph and commented on the many delicate and deep a priori estimates in by book.

References

[1] E. DiBenedetto and A. Friedman, *Hölder estimates for nonlinear degenerate parabolic systems,* J.Reine Angew. Math., **357**(1985), 1–32.

[2] Y. Z. Chen, *Hölder estimates for solutions of uniformly degenerate quasilinear parabolic equations,* Chinese Annals of Math., Vol.5, Ser. B, No.**4**(1984).

[3] G. M. Lieberman, *Interior gradient bounds for non- uniformly parabolic equations*, Indiana Univ. Math. J., **32**(1983), 579– 601.