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Note on the Maximal Coefficient of the Multinomial $(x_1+x_2+\cdots+x_k)^{n*}$

Zhu Ziqing
(Dept. of Math., Huazhong University of Science and Technology)

In a paper entitled "maximal Coefficient of the Multinomial $(x_1+x_2+\cdots+x_k)^n$" by Wu Qiqi [1], it was shown that the maximal coefficient $\bar{c}$ is given by the following cases:

(i) If $k \mid n, n/k = d \geq 1$, then $\bar{c} = n!/(d!)^k$;

(ii) If $k \mid n, n \equiv (\bmod k), 1 \leq r \leq k-1$ with $z = [\frac{n}{k}]$ (integer part of n/k), then $\bar{c} = n!/(z!)^{k-r}((z+1)!)^r$.

Here it may be noticed that $kd = n$ or $(k-r)z + r(z+1) = n$.

The original proof is unnecessarily complicated and diffuse. Actually the values $\bar{c}$ given by (i) and (ii) can be proved very simply.

Recall that the multinomial number is generally denoted by

$$\binom{n}{n_1,\cdots,n_k} = (n; n_1,\cdots,n_k) = \frac{n!}{n_1!\cdots n_k!}, \ (n_1+\cdots+n_k = n).$$

Thus for the case (i) with $kd = n$ it suffices to show that $(d!)^k \leq (n_1!)\cdots(n_k!)$, where $kd = d+\cdots+d = n_1+\cdots+n_k = n$.

One may say that $(n_1,\cdots,n_k)$ is a partition of n with k parts.

In particular, $(d,\cdots,d)$ is a partition of n with k parts. It is clear that the partition $(d,\cdots,d)$ can be transformed "step by step " to obtain $(n_1,\cdots,n_k)$ by removing units I's from one part into the other, successively.

For any two positive integers α and β with $\alpha \leq \beta$, it is evident that $\alpha!\beta! \leq (\alpha-1)!(\beta+1)!$. This implies that when $(d,\cdots,d)$ is transformed into $(n_1,\cdots,n_k)$, the product $(d!)^k = (d!)\cdots(d!) \leq (n_1!)\cdots(n_k!)$. The same argument applies to the partition of n with k parts:

$$(k-r)z + r(z+1) = \underbrace{z+\cdots+z}_{k-r} + \underbrace{(z+1)+\cdots+(z+1)}_{r} = n,$$

so that we have $(z!)^{k-r}((z+1)!)^r = z!\cdots z!(z+1)!\cdots(z+1)! \leq (n_1!)\cdots(n_k!)$ and the case (ii) is also justified.

References

[1] Wu Qiqi, *Maximal coefficient of the multinomail* $(x_1x_2\cdots x_k)^n$, SEA Bull. Math. Vol.**15**, No.**1**(1991), 77- 82.

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