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On Extremes of Determinants and Traces and Optimal Properties of Canonical Variables*

Liu Shuangzhe, *Li Cunjuan*
(Northeastern University, Shenyang 110006, China)

Abstract Extremes of quadratic forms have been discussed in [1] and [3]. Some related topics can be found in [2]. Some extremes of determinants and optimal properties of canonical variables were obtained in [1]. Here discussed are some extensions to [1].

1. Determinants and Traces

Theorem 1 *For compatible matrices T and W, non-negative definite (n.n.d.) matrix A and positive definite (p.d.) matrix D,*

(1) $|T'AW|^2 \leq |T'AT||W'AW|$, *with equality iff* $T'AT$ *or* $W'AW$ *are singular, or* $AT = AWQ$ *for some nonsingular matrix* Q.

(2) $|T'W|^2 \leq |T'DT||W'D^{-1}W|$, *with equality iff* $T'DT$ *or* $W'D^{-1}W$ *are singular, or* $DT = WQ$ *for some nonsingular matrix* Q.

Theorem 2 *For compatible n.n.d. matrix A, p.d. matrix D, $n \times k$ full column rank matrix $T, k \times k$ identity matrix $I_k, \lambda_1(\cdot) \geq \cdots \geq \lambda_n(\cdot)$.*

(1) $$\prod_{i=n-k+1}^{n} \lambda_i(D^{-1}A) \leq |T'AT|/|T'DT|,$$

(2) $$\inf_{T'DT=I_k}|T'AT| = \prod_{i=n-k+1}^{n} \lambda_i(D^{-1}A).$$

Theorem 3 *For compatible matrices T and W, n.n.d. matrix A and p.d. matrix D,*

(1) $(trT'AW)^2 \leq (trT'AT)(trW'AW)$, *with equality iff the matrices* AT *and* AW *are proportional,*

(2) $(trT'W)^2 \leq (trT'DT)(trW'D^{-1}W)$, *with equality iff the matrices* DT *and* W *are proportional,*

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(3) $(trT'AW)^2 \leq (trT'ATW'AW)$, *with equality iff* $ATW'A = AWT'A$,

(4) $(trT'W)^2 \leq (trT'DTW'D^{-1}W)$, *with equality iff* $DTW' = WT'D$.

Theorem 4 *For* $n \times n$ *symmetric* A, *p.d. matrix* D *and* $n \times k$ *matrix* $T \neq 0$,

$$\lambda_n(D^{-1}A) \leq (trT'AT)/(trT'DT) \leq \lambda_1(D^{-1}A).$$

Theorem 5 *For* $n \times m$ *matrix* $A, n \times k$ *matrix* $T \neq 0, m \times k$ *matrix* $W \neq 0, n \times n$ *p.d. matrix* D_1 *and* $m \times m$ *p.d. matrix* D_2,

$$\lambda_n(D_1^{-1}AD_2^{-1}A') \leq (trT'AW)^2/[(trT'D_1T)(trW'D_2W)] \leq \lambda_1(D_1^{-1}AD_2^{-1}A'),$$

where the first relationship also needs that the matrices D_1T *and* AW *(or* $A'T$ *and* D_2W*) are proportional.*

Theorem 6 *For* $n \times n$ *symmetric matrix* A, *p.d. matrix* $D, n \times k$ *full column rank matrix* T,

(1) $\sup_{T'DT=Ik} trT'AT = \sum_{i=1}^{k} \lambda_i(D^{-1}A)$,

(2) $\inf_{T'DT=Ik} trT'AT = \sum_{i=n-k+1}^{n} \lambda_i(D^{-1}A)$.

(3) $\sup_{T'DT=Ik} tr(T'AT)^{-1} = \sum_{i=n-k+1}^{n} \lambda_i^{-1}(D^{-1}A)$, *for p.d. matrix* A,

(4) $\inf_{T'DT=Ik} tr(T'AT)^{-1} = \sum_{i=1}^{k} \lambda_i^{-1}(D^{-1}A)$, *for p.d. matrix* A.

Theorem 7 *For* $n \times m$ *matrix* A, *full column rank* $n \times k$ *matrix* $T, m \times k$ *matrix* $W, n \times n$ *p.d. matrix* D_1 *and* $m \times m$ *p.d. matrix* D_2,

(1) $\sup_{T'D_1T=Ik, W'D_2W=Ik} (trT'AW)^2 = k \sum_{i=1}^{k} \lambda_i(D_1^{-1}AD_2^{-1}A')$,

(2) $\inf_{T'D_1T=Ik, W'D_2W=Ik} (trT'AW)^2 \leq k \sum_{i=n-k+1}^{n} \lambda_i(D_1^{-1}AD_2^{-1}A')$, *with equality if the matrices* D_1T *and* AW *(or* $A'T$ *and* D_2W*) are proportional.*

Theorem 8 *For any* $n \times n$ *matrix* A, *full column rank* $n \times k$ *matrices* T *and* $W, n \times n$

p.d. matrices D_1 and D_2,

$$\sup_{T'D_1T=Ik,W'D_2W=Ik} tr(T'AW)^2 \le \sum_{i=1}^{k}\lambda_i(D_1^{-1}AD_2^{-1}A'),$$
$$\inf_{T'D_1T=Ik,W'D_2W=Ik} tr(T'AW)^2 \le \sum_{i=n-k+1}^{n}\lambda_i(D_1^{-1}AD_2^{-1}A'),$$

both with equality if $AWT'D_1 = D_1TW'A'$, or $D_2WT'A = A'TW'D_2$.

Note Specifically we can take D, D_1 and D_2 as compatible identity matrices respectively in the above theorems.

2. Statistical Applications

Let $z' = (x', y'), x = (x_1, \cdots, x_n)'$ and $y = (y_1, \cdots, y_m)'$ be two random vector with the expectation $E(z) = 0$ and the variance matrix $D(z) = \Sigma$ with st^{th} position $\Sigma_{st}, s, t = 1, 2.\Sigma_{11}$ and Σ_{22} be non-singular and rank$(\Sigma_{12}) = r \ge k$.

Theorem 9 *Let T and W be $n \times k$ and $m \times k$ matrices with $T'T = W'W = I_k$. Then*

(1) $$trD(T'\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1/2}y) \le trD(P_k'\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1/2}y),$$

$$trD(W'\Sigma_{22}^{-1/2}\Sigma_{21}\Sigma_{11}^{-1/2}x) \le trD(Q_k'\Sigma_{22}^{-1/2}\Sigma_{21}\Sigma_{11}^{-1/2}x),$$

where P_k and Q_k satisfy the following equations

$$\begin{aligned} P_k'\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1/2}P_k &= diag(\lambda_1, \cdots, \lambda_k), \\ Q_k'\Sigma_{22}^{-1/2}\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1/2}Q_k &= diag(\lambda_1, \cdots, \lambda_k), \\ P_k'P_k &= Q_k'Q_k = I_k, \end{aligned}$$

where $\lambda_1 \ge \cdots \ge \lambda_k > 0$ are the eigenvalues of $\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1/2}$ and therefore of $\Sigma_{22}^{-1/2}\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1/2}$, $P_k := (p_1, \cdots, p_k), Q_k := (q_1, \cdots, q_k), p_i$ and q_i are the eigenvectors of $\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1/2}$ and $\Sigma_{22}^{-1/2}\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1/2}$ respectively, associated with $\lambda_i(i = 1, \ldots, k)$.

(2) $$[k^{-1}tr\,Cov(T'\Sigma_{11}^{-1/2}x, W'\Sigma_{22}^{-1/2}y)]^2 \le k^{-1}\Sigma_{i=1}^{k}\lambda_i(\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}).$$

Theorem 10 *Let T and W be $n \times k$ and $m \times k$ matrices with $T'\Sigma_{11}T = W'\Sigma_{22}W = I_k$. Then*

(1) $$trD(T'\Sigma_{12}\Sigma_{22}^{-1}y) \le trD(A'\Sigma_{12}\Sigma_{22}^{-1}y),$$

$$trD(W'\Sigma_{21}\Sigma_{11}^{-1}x) \le trD(B'\Sigma_{21}\Sigma_{11}^{-1}x),$$

here A and B satisfy the following equations

$$\begin{aligned} A'\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}A &= diag(\lambda_1,\cdots,\lambda_k), \\ B'\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}B &= diag(\lambda_1,\cdots,\lambda_k), \\ A'\Sigma_{11}A &= B'\Sigma_{22}B = I_k. \end{aligned}$$

(2) $$[k^{-1}trCov(T'x,W'y)]^2 \le k^{-1}\Sigma_{i=1}^{k}\lambda_i(\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}).$$

Theorem 9 and 10 are related to the canonical variables and generalized correlation coeficients, see [1]. For other applications, see e.g. [2] and [3].

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有关行列式和迹的极值以及典则变量的最优性的补注

刘双喆　　李村涓

(东北大学，沈阳 110006)

摘　要

二次型极值已被一些文献诸如[1]和[3]讨论，一些相近的题目可在文献[2]中见到. 文献[1]得到与二次型有关的行列式极值和典则变量的最优性质，本文对[1]给出补注.