

关于双顶边星图的优美性*

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对于一简单图 $G=(V, E)$, 若对于每一个 $v \in V$, 存在一个整数 $l(v)$ (称为顶点 v 的标号) 使满足:

(a) $\forall u, v \in V$, 若 $u \neq v$, 则 $l(u) \neq l(v)$;

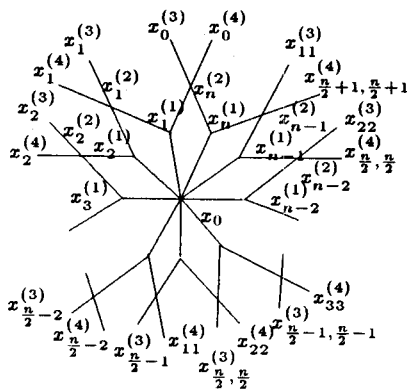
(b) $\max\{l(v) | v \in V\} = |E|$;

(c) $\forall e', e'' \in E$, 若 $e' \neq e''$, 则 $l'(e') \neq l'(e'')$, 这里 $l'(e)$ 定义为 $|l(u) - l(v)|$, 若 $e = uv$, 则称 G 为优美图 (graceful graph).

在文献[1]中, C. Hoede and H. Kuiper 证明了所有的轮都是优美图; 文献[2]又指出了所有的齿轮图也是优美图. 本文将给出在齿轮图每个齿的顶端加上两条长度为1的边所得的图亦为优美图.

定义 在齿轮图 $\tilde{W}_n$ 的每个齿的顶端加上两条长度为1的边后所得的图称为双顶边星图. 记为 $\tilde{W}_n^*$.

下面分别讨论当 n 为偶数或奇数时, $\tilde{W}_n^*$ 的优美性.



(图 1)

引理 1 当 n 为偶数时, $\tilde{W}_n^*$ 是优美图.

证明 当 $n \leq 6$ 时容易给出 $\tilde{W}_n^*$ 的直观优美标号图.

当 $n \geq 8$ 时定义顶点标号 $l(\tilde{W}_n^*)$ 如图 1 所示:

$$l(x_0) = 0; l(x_i^{(1)}) = 5n - (i - 1), i = 1, 2, 3, \dots, n.$$

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$$l(x_i^{(2)}) = n + i - 1, \quad i = 1, 2, 3, \dots, n-1.$$

$$l(x_n^{(2)}) = l(x_n^{(1)}) - 3; \quad l(x_0^{(3)}) = l(x_n^{(2)}) - 1.$$

$$l(x_i^{(3)}) = n - (i + 4), \quad i = 1, 2, 3, \dots, \frac{n}{2} - 1.$$

$$l(x_{\frac{n}{2}}^{(3)}) = 2n + (i - 1), \quad i = 2, 3, \dots, \frac{n}{2}.$$

$$l(x_{11}^{(3)}) = l(x_n^{(2)}) + 2.$$

$$l(x_0^{(4)}) = l(x_0^{(3)}) - 1.$$

$$l(x_i^{(4)}) = \frac{5n}{2} + 2j, \quad j = 2, 3, 4, \dots, \frac{n}{2} - 1; \quad i = j - 1.$$

$$l(x_{\frac{n}{2}}^{(4)}) = \frac{5n}{2} + (2i - 1), \quad i = 1, 2, 3, 4, \dots, \frac{n}{2} + 1.$$

特别指出, 当 $n = 8$ 时定义: $l(x_0^{(3)}) = 31, l(x_0^{(4)}) = 28$.

当 $n = 10$ 时定义: $l(x_0^{(3)}) = 20, l(x_0^{(4)}) = 36; l(x_{\frac{n}{2}+1, \frac{n}{2}+1}^{(4)}) = l(x_{66}^{(4)}) = 19$.

(a) 易见 l 是由 $V(\tilde{W}_n^*)$ 到 $\{l(x) | x \in V(\tilde{W}_n^*)\} = \{0; n, n+1, \dots, 2n-2; n-5, n-6, \dots, \frac{n}{2}-3; 2n+1, 2n+2, \dots, \frac{5n}{2}-1; \frac{5n}{2}+1, \frac{5n}{2}+3, \dots, \frac{5n}{2}+(n+1); \frac{5n}{2}+4, \frac{5n}{2}+6, \dots, \frac{5n}{2}+(n-2); 4n-4; 4n-3; 4n-2; 4n; 4n+1, 4n+2, \dots, 5n\}$ 的 1-1 映射. 因而, $\forall x', x'' \in V(\tilde{W}_n^*)$, 若 $x' \neq x''$, 则 $l(x') \neq l(x'')$.

(b) 显然, $\max\{l(x) | x \in V(\tilde{W}_n^*)\} = l(x_1^{(1)}) = 5n = |E(\tilde{W}_n^*)|$

(c) 分下面八步考虑:

$$(1) \quad I_1 = l'(x_0^{(3)} x_n^{(2)}) = 1; \quad I_2 = l'(x_n^{(2)} x_n^{(1)}) = 3; \quad I_3 = l'(x_0^{(4)} x_n^{(2)}) = 2.$$

(2) $l'(x_{\frac{n}{2}}^{(3)} x_{n-i}^{(2)}) = 2i$, 故当 i 取遍 $\{2, 3, 4, \dots, \frac{n}{2}\}$ 时, $l'(x_{\frac{n}{2}}^{(3)} x_{n-i}^{(2)})$ 取遍:

$$I_4 = \{4, 6, 8, \dots, n\}$$

(3) $l'(x_i^{(3)} x_i^{(2)}) = 2i + 3$, 故当 i 取遍 $\{1, 2, 3, \dots, \frac{n}{2} - 1\}$ 时, $l'(x_i^{(3)} x_i^{(2)})$ 取遍:

$$I_5 = \{5, 7, 9, \dots, n+1\}.$$

$$I_6 = l'(x_n^{(2)} x_1^{(1)}) = n+2; \quad I_7 = l'(x_{11}^{(4)} x_{\frac{n}{2}-1}^{(2)}) = n+3.$$

(4) $l'(x_{\frac{n}{2}}^{(4)} x_j^{(2)}) = (n+2) + i, \quad i = 2, 3, 4, \dots, \frac{n}{2} + 1; \quad j = \frac{n}{2} + (i-2)$. 故当 i 取遍 $\{2, 3, 4, \dots, \frac{n}{2} + 1\}$ 时, $l'(x_{\frac{n}{2}}^{(4)} x_j^{(2)})$ 取遍:

$$I_8 = \{n+4, n+5, n+6, \dots, \frac{3n}{2} + 3\}.$$

(5) $l'(x_i^{(4)} x_i^{(2)}) = \frac{3n}{2} + j, \quad j = 4, 5, 6, \dots, \frac{n}{2} + 1; \quad i = j - 3$. 故当 i 取遍 $\{4, 5, 6, \dots, \frac{n}{2} + 1\}$ 时, $l'(x_i^{(4)} x_i^{(2)})$ 取遍:

$$I_9 = \{2n+1, 2n, 2n-1, \dots, \frac{3n}{2} + 4\}.$$

$$I_{10} = l'(x_{n-1}^{(2)} x_{11}^{(3)}) = 2n+2.$$

(6) $\nu(x_j^{(2)}x_i^{(1)}) = 4n - (2i - 3), i = 2, 3, 4, \dots, n; j = i - 1$. 故当 i 取遍 $\{2, 3, 4, \dots, n\}$ 时, $\nu(x_j^{(2)}x_i^{(1)})$ 取遍:

$$I_{11} = \{2n + 3, 2n + 5, \dots, 4n - 5, 4n - 3, 4n - 1\}.$$

(7) $\nu(x_i^{(1)}x_i^{(2)}) = 4n - 2(i - 1)$, 故当 i 取遍 $\{1, 2, 3, \dots, n - 1\}$ 时, $\nu(x_i^{(1)}x_i^{(2)})$ 取遍:

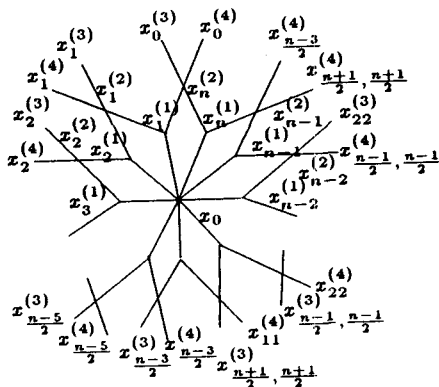
$$I_{12} = \{4n, 4n - 2, 4n - 4, \dots, 2n + 4\}.$$

(8) $\nu(x_0x_i^{(1)}) = 5n - (i - 1)$, 故当 i 取遍 $\{1, 2, 3, 4, \dots, n\}$ 时, $\nu(x_0x_i^{(1)})$ 取遍:

$$I_{13} = \{4n + 1, 4n + 2, \dots, 5n - 2, 5n - 1, 5n\}.$$

由 (1) - (8) 即可看出, 当 e 取遍 $E(\tilde{W}_n^*)$ 时, $\nu(e)$ 取遍: $I = I_1 + I_2 + I_3 + \dots + I_{13} = \{1, 2, 3, \dots, 5n\}$, 又因 $|E(\tilde{W}_n^*)| = 5n = |I|$, 故 ν 是一个由 $E(\tilde{W}_n^*)$ 到 I 的一个 1-1 映射, 从而, $\forall e', e'' \in E(\tilde{W}_n^*)$, 若 $e' \neq e''$, 则有 $\nu(e') \neq \nu(e'')$.

由 (a) - (c) 便知 $\tilde{W}_n^*$ 是优美图.



(图 2)

引理 2 当 n 为奇数时, $\tilde{W}_n^*$ 是优美图.

证明 当 $n \leq 7$ 时容易给出 $\tilde{W}_n^*$ 的直观优美标号图.

当 $n \geq 9$ 时定义顶点标号 $l(\tilde{W}_n^*)$ 如图 2 所示:

$$l(x_0) = 0; l(x_i^{(1)}) = 5n - (i - 1), i = 1, 2, \dots, n.$$

$$l(x_i^{(2)}) = n + (i - 1), i = 1, 2, 3, \dots, n - 1.$$

$$l(x_n^{(2)}) = l(x_n^{(1)}) - 3; l(x_0^{(3)}) = l(x_n^{(2)}) + 1.$$

$$l(x_i^{(3)}) = n - (i + 4), i = 1, 2, 3, \dots, \frac{n-3}{2}.$$

$$l(x_u^{(3)}) = 2n + (i - 1), i = 2, 3, 4, \dots, \frac{n+1}{2}.$$

$$l(x_{11}^{(3)}) = \frac{7n+3}{2}.$$

$$l(x_0^{(4)}) = l(x_0^{(3)}) + 1.$$

$$l(x_i^{(4)}) = \frac{5n+1}{2} + 2j, j = 2, 3, 4, \dots, \frac{n-1}{2}; i = j - 1.$$

$$l(x_u^{(4)}) = \frac{5n+1}{2} + (2i - 1), i = 1, 2, 3, 4, \dots, \frac{n+1}{2}.$$

类似于引理 1, 可以证明 $\hat{W}_n^*$ 是优美图.

由引理 1 及引理 2 使得:

定理: 所有双顶边星图 $\hat{W}_n^*$ 都是优美图.

基于上面定理, 我们不妨把在齿轮图每个齿的顶端加上 n 条长度为 1 的边后所得的图, 叫做 n 顶边星图. 对此, 最后提出如下猜想:

猜想: 所有 n 顶边星图都是优美图.

参 考 文 献

- [1] C. Hoede and H. Kuiper, *All wheels are graceful*, Utilitas Math, 14(1978), 311.
- [2] 马克杰、冯成进, 关于齿轮图的优美性, 数学的实践与认识, 4(1984), 72—73.

On the Gracefulness of the Star Graph with Two Top Sides

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Abstract

In this paper, we have defined the star graph with two top sides and the star graph with n top sides. It is shown that all the star graphs with two top sides are graceful graphs. Finally, we have proposed that all the star graphs with n top sides are graceful graphs.