三维涡度方程双向周期问题的拟谱—差分解法

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摘　要

本文构造了三维涡度方程双向周期问题的Fourier拟谱—差分格式，其数值解满足半离散守恒律. 文中分析了格式的广义稳定性和收敛性. 数值例子表明这类格式的优越性.

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Pseudospectral-Finite Difference Method for Three-Dimensional Vorticity Equation with Bilaterally Periodic Boundary Conditions*

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Abstract A Fourier pseudospectral-finite difference scheme is proposed for three-dimensional vorticity equation with bilaterally periodic boundary conditions. The numerical solution possesses semi-discrete conservation. The generalized stability and convergence are analyzed.

Keywoods Three-dimensional vorticity eqution, bilaterally periodic boundary condition, pseudospectral-finite difference scheme.

I. Introduction

During the past fifteen years, spectral and pseudospectral methods developed rapidly, e.g., see [1–2]. Since both of them have the accuracies of "infinte" order and could often be evaluated explicitly, they have been applied successfully to numerical simulations in fluid dynamics (see [3–6]). For nonlinear problems, the pseudospectral method is easier to be implemented. But it is not as stable as spectral method due to the aliasing interaction. There have been two main filtering techniques to remedy this deficiency. The first was proposed by Kuo Penyu, based on Bochner summation (see [6–8]). The second was given by Woodward, Collela and Vandeven (see [9–10]).

In studying the boundary-layer stability, the unsteady separation, the flow past a suddenly heated vertical plate and related topics, we meet bilaterally periodic problems. Such problems are usually treated in two efficient ways. One is the application of mixed spectral method, such as Fourier-Chebyshev approximation (see [11]) or Fourier-Legendre approximation. The other is to use spectral-difference method or spectral-finite element method (see [12– 17]). Clearly it is more natural to use the former one provided that the domain is rectangular. Moreover, for saving computational time and dealing with nonlinear terms more easily, we prefer to adopt pseudospectral-finite difference method with filtering technique, e.g., see [18–20].

This paper is devoted to pseudospectral-finite difference scheme for three- dimensional vorticity equation with bilaterally periodic boundary conditions. Let $x = (x_1, x_2, x_3)^*$ and Ω be a rectangular domain, say that

$$\Omega = Q \times I, \quad Q = \{(x_1, x_2) | 0 < x_1, x_2 < 2\pi\}, \quad I = \{x_3 | 0 < x_3 < 1\}.$$

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We denote by $\xi(x,t)$ and $\psi(x,t)$ the vorticity vector and the stream vector respectively. Their components are $\xi^{(p)}(x,t)$ and $\psi^{(p)}(x,t), 1 \le p \le 3$. $\nu > 0$ is the kinetic viscosity. Then the three-dimensional vorticity equation is of the form

$$\begin{cases} \frac{\partial \xi}{\partial t} + ((\nabla \times \psi)\cdot \nabla)\xi - (\xi \cdot \nabla)(\nabla \times \psi) - \nu \nabla^2 \xi = f_1, & \text{in } \Omega \times (0,T], \\ -\nabla^2 \psi = \xi + f_2, & \text{in } \Omega \times (0,T], \\ \xi(x,0) = \xi_0(x), & \text{in } \overline{\Omega}, \end{cases} \tag{1.1}$$

where f_l and ξ_0 are given functions with the components $f_l^{(p)}$ and $\xi_0^{(p)}, 1 \le p \le 3$. Also suppose that all functions mentioned above have the period 2π for the variables x_1 and x_2. Besides

$$\begin{cases} \psi = \chi \\ a\frac{\partial \xi}{\partial n} + b\xi = g, \qquad \text{for } x_3 = 0, 1, \end{cases} \tag{1.2}$$

where χ and g are given functions, a and b are non-negative constants.

An outline of this paper is as follows. We construct the scheme in Section II and present numerical results in Section III. Since we adopt skew-symmetric decomposion of the convection term, not only the numerical solution possesses a semi-discrete conservation, but also the main error terms depend only on the errors on boundary. We list some lemmas in Section IV and then analyze the generalized stability and convergence in Section V and Section VI. In particular, all error estimations include the influence of errors on the boundary. As we know, the errors of boundary values effect the accuracy seriously in calculation. But most of papers in computational dynamics neglected this important fact.

II. The Scheme and the Conservation

We first introduce some notations for Fourier approximation. Let $l = (l_1, l_2), l_q$ being integer. $|l| = (l_1^2 + l_2^2)^{\frac{1}{2}}$ and set

$$\tilde{V}_N = \text{Span}\{e^{i(l_1x_1 + l_2x_2)} \mid |l| \le N\}.$$

The subset of real valued functions in $\tilde{V}_N$ is denoted by V_N. Furthermore let P_N be the orthogonal projection from $L^2(Q)$ onto V_N, i.e., for any $u \in L^2(Q)$,

$$\int_Q (P_N u - u)v dx_1 dx_2 = 0, \quad \forall v \in V_N.$$

Also, we take the set of interpolation nodes as

$$Q_N = \{(x_1,x_2)/x_1 = \frac{2\pi j_1}{2N+1}, x_2 = \frac{2\pi j_2}{2N+1}, 0 \le j_q \le 2N, q = 1,2\}.$$

Let P_c be the interpolation operator from $C(\bar{Q})$ onto V_N such that for any $u \in C(\bar{Q})$,

$$P_c u(x_1,x_2) = u(x_1,x_2), \quad (x_1,x_2) \in Q_N.$$

We next consider the finite difference approximation. Let M be a non- negative integer, $h=\frac{1}{M}$ and $I_h=\{x_3|x_3=jh, 1\le j\le M-1\}, \bar{I}_h=I_h\cup\{0\}\cup\{1\}$. For any $u\in C(\bar{I})$,

$$u_{x_3}(x_3)=\frac{1}{h}(u(x_3+h)-u(x_3)),\ u_{\bar{x}_3}(x_3)=u_{x_3}(x_3-h),\ u_{\hat{x}_3}(x_3)=\frac{1}{2}u_{x_3}(x_3)+\frac{1}{2}u_{\bar{x}_3}(x_3).$$

For the discretization of time t, let τ be the mesh spacing, and

$$\begin{aligned} S_\tau &= \{t|t=k\tau, k=0,1,2,\ldots\},\ \ \bar{S}_\tau=S_\tau\cup\{0\},\\ u_t(t) &= \frac{1}{\tau}(u(t+\tau)-u(t)). \end{aligned}$$

Now we turn to the mixed approximation. Let

$$\Omega_h=Q\times I_h,\ \ \bar{\Omega}_h=\bar{Q}\times\bar{I}_h,\ \Gamma_h=\{x|x\in\bar{\Omega}_h, x_3=0\ \text{ or } 1\}.$$

Hereafter $u(x,t), v(x,t)$ and $w(x,t)$ are vectors in $\mathbf{R}^3$ with the period 2π for the variables x_1 and x_2. The meanings of $u_{x_3}(x,t), u_{\bar{x}_3}(x,t), u_{\hat{x}_3}(x,t)$ and $u_t(x,t)$ are clear. Also define

$$\Delta u(x,t)=\frac{\partial^2 u}{\partial x_1^2}(x,t)+\frac{\partial^2 u}{\partial x_2^2}(x,t)+u_{x_3\bar{x}_3}(x,t).$$

On the other hand, we still use the notations P_N and P_c for $u(x,t)$, for instance,

$$P_c u(x,t)=u(x,t),\ \ \text{for } (x_1,x_2)\in Q_N, x_3\in I_h, t\in S_\tau.$$

As in [6–8], we adopt the restrain operator $R=R(r)$ for the improvement of stability. It means that if $u(x,t)=\sum_{|l|\le N}u_l(x_3,t)e^{i(l_1x_1+l_2x_2)}$, then

$$Ru(x,t)=\sum_{|l|\le N}(1-(\frac{l}{N})^r)u_l(x_3,t)e^{i(l_1x_1+l_2x_2)},\ \ r\ge 1.$$

It is commonly admitted that a reasonable scheme should simulate some properties of continuous model. Indeed the solution of (1.1) possesses the conservation

$$\begin{aligned} &\|\xi(t)\|^2_{L^2(\Omega)}+\int_0^t[2\nu|\xi(y)|^2_{H^1(\Omega)}+2\nu(\xi(0,y),\frac{\partial\xi}{\partial x_3}(0,y))_{L^2(Q)}\\ &\quad -2\nu(\xi(1,y),\frac{\partial\xi}{\partial x_3}(1,y))_{L^2(Q)}-(\frac{\partial\psi^{(2)}}{\partial x_1}(0,y)\\ &\quad -\frac{\partial\psi^{(1)}}{\partial x_2}(0,y),\xi^2(0,y))_{L^2(Q)}+(\frac{\partial\psi^{(2)}}{\partial x_1}(1,y)-\frac{\partial\psi^{(1)}}{\partial x_2}(1,y),\xi^2(1,y))_{L^2(Q)}\\ &\quad -2((\xi(y)\cdot\nabla)(\nabla\times\psi(y)),\xi(y))_{L^2(Q)}]dy\\ &\ =\|\xi_0\|^2_{L^2(\Omega)}+2\int_0^t(f_1(y),\xi(y))_{L^2(\Omega)}dy. \end{aligned} \tag{2.1}$$

For simulating (2.1), we use the skew-symmetric decomposition as follows

$$((\nabla\times w)\cdot\nabla)v=\frac{1}{2}((\nabla\times w)\cdot\nabla)v+\frac{1}{2}\sum_{q=1}^{3}\frac{\partial}{\partial x_q}((\nabla\times w)\cdot e_q)v,$$

where $e_1=(1,0,0)^*, e_2=(0,1,0)^*, e_3=(0,0,1)^*$. We approach the term $\nabla\times w$ by $Z(w)$ with the components

$$Z^{(1)}(w)=\frac{\partial w^{(3)}}{\partial x_2}-w^{(2)}_{\hat{x}_3},Z^{(2)}(w)=w^{(1)}_{\hat{x}_3}-\frac{\partial w^{(3)}}{\partial x_1},Z^{(3)}(w)=\frac{\partial w^{(2)}}{\partial x_1}-\frac{\partial w^{(1)}}{\partial x_2}$$

and then approximate the term $((\nabla\times w)\cdot\nabla)v$ by $J(v,w)$,

$$\begin{aligned}
J(v,w) &= \frac{1}{2}J_1(v,w)+\frac{1}{2}J_2(v,w),\\
J_1(v,w) &= P_c(Z^{(1)}(w)\frac{\partial v}{\partial x_1}+Z^{(2)}(w)\frac{\partial v}{\partial x_2}+Z^{(3)}(w)v_{\hat{x}_3}),\\
J_2(v,w) &= \frac{\partial}{\partial x_1}(P_c(Z^{(1)}(w)v))+\frac{\partial}{\partial x_2}(P_c(Z^{(2)}(w)v))+(P_c(Z^{(3)}(w)v))_{\hat{x}_3}.
\end{aligned}$$

Besides the term $(\xi\cdot\nabla)(\nabla\times\psi)$ is approximated by

$$H(v,w)=P_c((v^{(1)}\frac{\partial}{\partial x_1}(Z(w))+v^{(2)}\frac{\partial}{\partial x_2}(Z(w))+v^{(3)}(Z(w))_{\hat{x}_3}).$$

Now let $\eta^{(N)}$ and $\varphi^{(N)}$ be the approximations to ξ and ψ. For any $x_3\in\bar{I}_h$ and $t\in S_\tau, \eta^{(N)},\varphi^{(N)}\in V_N^3$. Then the pseudospectral-finite difference scheme for (1.1) is

$$\begin{cases}
\eta_t^{(N)}(x,t)+RJ(R(\eta^{(N)}(x,t)+\delta\tau\eta_t^{(N)}(x,t)),\\
\qquad\qquad R\varphi^{(N)}(x,t))-RH(R\eta^{(N)}(x,t),R\varphi^{(N)}(x,t))\\
-\nu\Delta(\eta^{(N)}(x,t)+\delta\tau\eta_t^{(N)}(x,t))=P_cf_1(x,t),\\
-\Delta\varphi^{(N)}(x,t)=\eta^{(N)}(x,t)+P_cf_2(x,t),\\
\eta^{(N)}(x,0)=\eta^{(N)}(x)=P_c\xi_0(x),
\end{cases}\tag{2.2}$$

where σ and δ are parameters, $0\le\sigma,\delta\le 1$.

Before we check the conservation, we introduce some semi-discrete inner products and norms. Let

$$\begin{aligned}
(u(x_3,t),v(x_3,t))_Q &= \frac{1}{4\pi^2}\int_Q u(x,t)v(x,t)dx_1dx_2,\\
\|u(x_3,t)\|_Q^2 &= (u(x_3,t),u(x_3,t))_Q,\\
(u(x_1,x_2,t),v(x_1,x_2,t))_{I_h} &= h\sum_{x_3\in I_h}u(x,t)v(x,t),\\
\|u(x_1,x_2,t)\|_{I_h}^2 &= (u(x_1,x_2,t),u(x_1,x_2,t))_{I_h},
\end{aligned}$$

$$\begin{aligned}(u(t),v(t)) &= h\sum_{x_3\in I_h}(u(x_3,t),v(x_3,t))_Q,\\ \|u(t)\|^2 &= (u(t),u(t)), \|u(t)\|_{l^4}^4 = \|u^2(t)\|^2,\\ |u(t)|_1^2 &= \|\frac{\partial u}{\partial x_1}(t)\|^2 + \|\frac{\partial u}{\partial x_2}(t)\|^2 + \frac{1}{2}\|u_{\bar{x}_3}(t)\|^2 + \frac{1}{2}\|u_{x_3}(t)\|^2,\\ \|u(t)\|_1^2 &= |u(t)|_1^2 + \|u(t)\|^2\end{aligned}$$

and

$$\begin{aligned}|u(t)|_2^2 = \ & |\frac{\partial u}{\partial x_1}(t)|_1^2 + |\frac{\partial u}{\partial x_2}(t)|_1^2 + \frac{1}{2}\|u_{x_3\bar{x}_3}(t)\|^2\\ &+ \frac{h}{4}\sum_{2h\le x_3\le 1-h}\|u_{\bar{x}_3\bar{x}_3}(x_3,t)\|_Q^2 + \frac{h}{4}\sum_{h\le x_3\le 1-2h}\|u_{x_3x_3}(x_3,t)\|_Q^2,\\ &+ \|(\frac{\partial u}{\partial x_1}(t))_{\bar{x}_3}\|^2 + \|(\frac{\partial u}{\partial x_1}(t))_{x_3}\|^2 + \|(\frac{\partial u}{\partial x_2}(t))_{\bar{x}_3}\|^2 + \|(\frac{\partial u}{\partial x_2}(t))_{x_3}\|^2.\end{aligned}$$

Assume that $u,v,w\in V_N^3$ for any $x_3\in I_h$ and $t\in S_\tau$. Then (see [7])

$$(P_c(uv)(t),w(t)) = (u(t),P_c(vw)(t)). \tag{2.3}$$

It can be verified that

$$\begin{aligned}(u(t),v_{\hat{x}_3}(t)) + (u_{\hat{x}_3}(t),v(t)) = \ & \frac{1}{2}[(u(1,t),v(1-h,t))_Q + (u(1-h,t),v(1,t))_Q\\ &-(u(h,t),v(0,t))_Q - (u(0,t),v(h,t))_Q].\end{aligned} \tag{2.4}$$

By integrating by parts and (2.3), (2.4), we have

$$\begin{aligned}&(u(t),J(v(t),w(t))) + (v(t),J(u(t),w(t)))\\ &= \frac{1}{2}(v_{\hat{x}_3}(t),P_c(Z^{(3)}(w(t))u(t))) + \frac{1}{2}(v(t),(P_c(Z^{(3)}(w(t))u(t)))_{\hat{x}_3})\\ &\quad+\frac{1}{2}(u_{\hat{x}_3}(t),P_c(Z^{(3)}(w(t))v(t))) + \frac{1}{2}(u(t),(P_c(Z^{(3)}(w(t))v(t)))_{\hat{x}_3})\\ &= \frac{1}{2}A(u(t),v(t),w(t)) + \frac{1}{2}A(v(t),u(t),w(t)),\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}A(u(t),v(t),w(t)) = \ & \frac{1}{2}[(u(1,t),P_c(Z^{(3)}(w(1-h,t))v(1-h,t)))_Q\\ &+ (u(1-h,t),P_c(Z^{(3)}(w(1,t))v(1,t)))_Q - (u(h,t),P_c(Z^{(3)}(w(0,t))v(0,t)))_Q\\ &- (u(0,t),P_c(Z^{(3)}(w(h,t))v(h,t)))_Q].\end{aligned}$$

Similary, we can prove that

$$(u(t),v_{x_3\bar{x}_3}(t)) + \frac{1}{2}(u_{\bar{x}_3}(t),v_{\bar{x}_3}(t)) + \frac{1}{2}(u_{x_3}(t),v_{x_3}(t)) = B(u(t),v(t)), \tag{2.6}$$

where

$$B(u(t),v(t)) = \frac{1}{2}(u(1,t)+u(1-h,t),v_{\bar{x}_3}(1,t))_Q - \frac{1}{2}(u(h,t)+u(0,t),v_{x_3}(0,t))_Q.$$

Furthermore by using Green's formula and (2.6), we have

$$\begin{aligned}(u(t),\Delta v(t)) &+ (\frac{\partial u}{\partial x_1}(t),\frac{\partial v}{\partial x_1}(t)) + (\frac{\partial u}{\partial x_2}(t),\frac{\partial v}{\partial x_2}(t)) \\ &+ \frac{1}{2}(u_{\bar{x}_3}(t),v_{\bar{x}_3}(t)) + \frac{1}{2}(u_{x_3}(t),v_{x_3}(t)) = B(u(t),v(t))\end{aligned} \tag{2.7}$$

and thus $(u(t),\Delta u(t)) + |u(t)|_1^2 = B(u(t),u(t))$. If in addition $u(x,t)=0$ on $\Gamma_h \times S_\tau$, then

$$(u(t),\Delta u(t)) + |u(t)|_1^2 + S(u(t)) = 0, \tag{2.8}$$

where $S(u(t)) = \frac{1}{2h}\|u(1-h,t)\|_Q^2 + \frac{1}{2h}\|u(h,t)\|_Q^2$.

We now check the conservation. Letting $\delta=\sigma=\frac{1}{2}$ and taking the inner product with $\eta^{(N)}(x,t)+\eta^{(N)}(x,t+\tau)$ in the first formula of (2.2). We have from (2.5) and (2.8) that

$$\begin{aligned}&\|\eta^{(N)}(t)\|_t^2 + \frac{\nu}{2}|\eta^{(N)}(t)+\eta^{(N)}(t+\tau)|_1^2 \\ &\quad - \frac{\nu}{2}B(\eta^{(N)}(t)+\eta^{(N)}(t+\tau),\eta^{(N)}(t)+\eta^{(N)}(t+\tau)) \\ &\quad + \frac{1}{4}A(R(\eta^{(N)}(t)+\eta^{(N)}(t+\tau)),R(\eta^{(N)}(t)+\eta^{(N)}(t+\tau)),R\varphi^{(N)}(t)) \\ &\quad - (H(R\eta^{(N)}(t),R\varphi^{(N)}(t)),R(\eta^{(N)}(t)+\eta^{(N)}(t+\tau))) \\ &= (P_c f_1(t),\eta^{(N)}(t)+\eta^{(N)}(t+\tau)).\end{aligned}$$

Thus

$$\begin{aligned}&\|\eta^{(N)}(t)\|^2 + \frac{\nu\tau}{2}\sum_{\substack{y\in S_\tau\\ y\le t-\tau}}[|\eta^{(N)}(y)+\eta^{(N)}(y+\tau)|_1^2 \\ &\quad - B(\eta^{(N)}(y)+\eta^{(N)}(y+\tau),\eta^{(N)}(y)+\eta^{(N)}(y+\tau)) \\ &\quad + \frac{1}{4}A(R(\eta^{(N)}(y)+\eta^{(N)}(y+\tau)),R(\eta^{(N)}(y)+\eta^{(N)}(y+\tau)),R\varphi^{(N)}(y)) \\ &\quad - (H(R\eta^{(N)}(y),R\varphi^{(N)}(y)),R(\eta^{(N)}(y)+\eta^{(N)}(y+\tau)))] \\ &= \|\eta^{(N)}(0)\|^2 + \tau\sum_{\substack{y\in S_\tau\\ y\le t-\tau}}(P_c f_1(y),\eta^{(N)}(t)+\eta^{(N)}(t+\tau)).\end{aligned}$$

Clearly, this is a reasonable analogy of (2.1). Therefore (2.2) can give better numerical results.

III. Numerical Results

In this section, we present some numerical results of (2.2) with Dirichlet boundary condition for $x_3=0,1$. For convenience, let $\delta=\sigma=0$. We take the following test function

$$\begin{aligned}\xi^{(p)}(x,t) &= A_p \exp(B_p \sin(x_1+x_2+C_p x_3)+D_p t),\\ \psi^{(p)}(x,t) &= A_p \exp(D_p t)\sin x_1 \sin x_2 \sin C_p x_3, p=1,2,3.\end{aligned}$$

For comparison, we also consider a full finite difference scheme. Let $\bar{h}=\frac{2\pi}{2N+1}$ and $\Omega'_h = Q_N \times I_h$. The definition of $v_{x_q}, v_{\hat{x}_q}$ and $v_{\bar{x}_q}$ for $q=1,2$ are similar to that of $v_{x_3}, v_{\bar{x}_3}$ and $v_{\hat{x}_3}$. Also define

$$\bar{Z}^{(1)}(w)=w^{(3)}_{\hat{x}_2}-w^{(2)}_{\hat{x}_3}, \bar{Z}^{(2)}(w)=w^{(1)}_{\hat{x}_3}-w^{(3)}_{\hat{x}_1}, \bar{Z}^{(3)}(w)=w^{(2)}_{\hat{x}_1}-w^{(1)}_{\hat{x}_2},$$

$$\bar{J}(v,w)=\frac{1}{2}\sum_{p=1}^{3}(\bar{Z}^{(p)}(w)v_{\hat{x}_p}+(\bar{Z}^{(p)}(w)v)_{\hat{x}_p}),\ \bar{H}(v,w)=\sum_{p=1}^{3}v^{(p)}\sum_{\hat{x}_p}(w),$$

$$\bar{\Delta}v(x,t)=\sum_{p=1}^{3}v_{x_p\bar{x}_p}(x,t).$$

The full difference scheme for (1.1) is as follows (see [21])

$$\begin{cases} \eta^{\bar{h}}_t(x,t)+\bar{J}(\eta^{\bar{h}}(x,t),\varphi^{\bar{h}}(x,t))-\bar{H}(\eta^{\bar{h}}(x,t),\varphi^{\bar{h}}(x,t))-\nu\bar{\Delta}(\eta^{\bar{h}}(x,t))=f_1(x,t),\\ -\bar{\Delta}\varphi^{\bar{h}}(x,t)=\eta^{\bar{h}}(x,t)+f_2(x,t). \end{cases} \tag{3.1}$$

For measuring the errors, we define the following discrete norms

$$\begin{aligned}\tilde{E}(Z,t) &= \frac{(\sum_{p=1}^{3}\sum_{j_1=0}^{2N}\sum_{j_2=0}^{2N}\sum_{j_3=1}^{M-1}(Z^{(p)}(j_1\bar{h},j_2\bar{h},j_3h,t)-\xi^{(p)}(j_1\bar{h},j_2\bar{h},j_3h,t))^2)^{1/2}}{(\sum_{p=1}^{3}\sum_{j_1=0}^{2N}\sum_{j_2=0}^{2N}\sum_{j_3=1}^{M-1}(\xi^{(p)}(j_1\bar{h},j_2\bar{h},j_3h,t))^2)^{1/2}},\\ Z &= \eta^{(N)} \text{ or } \eta^{\bar{h}}.\end{aligned}$$

Table 1 and Table 2 show the numerical results of scheme (2.2) with different values of r and ν. In both cases, $A_l=D_l=0.1(1\le l\le 3), B_1=0.2, B_2=B_3=0.1, C_1=0.1, C_2=0.2, C_3=0.3, N=2, M=8, \tau=0.002$. We find that the suitable choice of r improves the stability. So the restrain operator $R(r)$ plays an important role in practical computation. In particular for small ν, scheme (2.2) with the restrain operator still provides better numerical result, see Table 3, where $A_l=B_l=C_l=0.1(1\le l\le 3), D_1=0.01, D_2=0.02, D_3=0.03, N=2, M=5, \tau=0.001$. Clearly scheme (2.2) gives good numerical results even for very small N. Table 4 shows the numerical results of scheme (2.2) with $r=2$ and scheme (3.1) for comparison. The calculation is carried out with the parameters $A_1=0.2, A_2=A_3=0.1, B_l=D_l=0.1, C_1=0.1, C_2=0.2, C_3=0.3, N=2, M=8, \tau=0.002$. It can be seen from Table 4 that pseudospectral-finite difference scheme (2.2) provides much better numerical results than full finite difference scheme (3.1).

Table 1. The error $\tilde{E}(\eta^{(N)},t), \nu=1.0$

t	$r=3$	$r=5$	$r=10$	$r=\infty$
0.2	$0.2171E-2$	$0.2369E-2$	$0.2482E-2$	$0.2490E-2$
1.0	$0.2357E-2$	$0.2589E-2$	$0.2718E-2$	$0.2726E-2$
2.0	$0.2560E-2$	$0.2823E-2$	$0.2967E-2$	$0.2977E-2$

Table 2. The error $\tilde{E}(\eta^{(N)},t), \nu = 0.1$

t	$r=2$	$r=5$	$r=10$	$r=\infty$
0.2	$0.1518E-1$	$0.1724E-1$	$0.1822E-1$	$0.1829E-1$
1.0	$0.1964E-1$	$0.2270E-1$	$0.2404E-1$	$0.2412E-1$
2.0	$0.2191E-1$	$0.2561E-1$	$0.2716E-1$	$0.2725E-1$

Table 3. The error $\tilde{E}(\eta^{(N)},t), \nu = 10^{-3}$

t	$r=5$	$r=10$	$r=20$	$r=\infty$
0.1	$0.6940E-2$	$0.7203E-2$	$0.7217E-2$	$0.7252E-2$
0.5	$0.3409E-1$	$0.3538E-1$	$0.3545E-1$	$0.3559E-1$
1.0	$0.6670E-1$	$0.6923E-1$	$0.6936E-1$	$0.6947E-1$

Table 4. The error $\tilde{E}(\eta^{(N)},t)$ and $\tilde{E}(\eta^{\bar{h}},t), \nu = 1.0$

t	$\tilde{E}(\eta^{(N)},t)$	$\tilde{E}(\eta^{\bar{h}},t)$
0.2	$0.2071E-2$	$0.1445E-1$
1.0	$0.2245E-2$	$0.1448E-1$
2.0	$0.2481E-2$	$0.1452E-1$

IV. Some Lemmas

In order to estimate the errors, we need some lemmas. We denote by C a positive constant which may be different in different cases. Let $H^\beta(Q)$ be the usual Sobolev space with the norm $\|\cdot\|_{H^\beta(Q)}$ and

$$H_p^\beta(Q) = \{v(x_1,x_2) \in H^\beta(Q)/v \text{ has the period } 2\pi \text{ for } x_1 \text{ and } x_2\}.$$

Lemma 1 *If $u(x,t) \in [L^2(Q)]^3$ for all $x_3 \in I_h$ and $t \in S_\tau$, then*

$$\begin{aligned} 2(u(x_3,t),u_t(x_3,t))_Q &= (\|u(x_3,t)\|_Q^2)_t - \tau\|u_t(x_3,t)\|_Q^2, \\ 2(u(t),u_t(t)) &= (\|u(t)\|^2)_t - \tau\|u_t(t)\|^2. \end{aligned}$$

Lemma 2 *If $u(x,t) \in V_N^3$ for all $x_3 \in \bar{I}_h$ and $t \in S_\tau$, then*

$$\begin{aligned} 2(u(t),\Delta u_t(t)) + (|u(t)|_1^2)_t - \tau|u_t(t)|_1^2 &= 2B(u(t),u_t(t)), \\ 2(u_t(t),\Delta u(t)) + (|u(t)|_1^2)_t - \tau|u_t(t)|_1^2 &= 2B(u_t(t),u(t)). \end{aligned}$$

Lemma 3 *If $u(x,t) \in V_N^3$ for all $x_3 \in \bar{I}_h$ and $t \in S_\tau$, then*

$$\|\frac{\partial u}{\partial x_q}(t)\|^2 \le N^2\|u(t)\|^2, \quad q=1,2,$$

$$\begin{aligned}
\|u_{\bar{x}_3}(t)\|^2 &\le \frac{4}{h^2}\|u(t)\|^2+\min(h\|u_{x_3}(0,t)\|_Q^2,\frac{2}{h}\|u(0,t)\|_Q^2),\\
\|u_{x_3}(t)\|^2 &\le \frac{4}{h^2}\|u(t)\|^2+\min(h\|u_{\bar{x}_3}(1,t)\|_Q^2,\frac{2}{h}\|u(1,t)\|_Q^2),\\
|u(t)|_1^2 &\le (2N^2+\frac{4}{h^2})\|u(t)\|^2+\frac{1}{2}\min(h\|u_{x_3}(0,t)\|_Q^2\\
&\quad +h\|u_{\bar{x}_3}(1,t)\|_Q^2,\frac{2}{h}\|u(0,t)\|_Q^2+\frac{2}{h}\|u(1,t)\|_Q^2).
\end{aligned}$$

Lemma 4 *Let $u(x,t)\in V_N^3$ for all $x_3\in\bar{I}_h$ and $t\in S_\tau$. If $u(x,t)=0$ on $\Gamma_h\times S_\tau$, then*

$$\|u(t)\|^2\le c(|u(t)|_1^2+S(u(t))).$$

Lemma 5 *Let $u(x,t)\in V_N^3$ for all $x_3\in\bar{I}_h$ and $t\in S_\tau$. If $u(x,t)=0$ on $\Gamma_h\times S_\tau$, then*

$$\|u(t)\|_2^2\le\|\Delta u(t)\|^2$$

Lemma 6 *If $u(x,t),v(x,t)\in V_N^3$ for all $x_3\in\bar{I}_h$ and $t\in S_\tau$, then*

$$\begin{aligned}
\|u(x_3,t)v(x_3,t)\|_Q^2 &\le (2N+1)^2\|u(x_3,t)\|_Q^2\|v(x_3,t)\|_Q^2,\\
\|u(x_1,x_2,t)v(x_1,x_2,t)\|_{I_h}^2 &\le \frac{1}{h}\|u(x_1,x_2,t)\|_{I_h}^2\|v(x_1,x_2,t)\|_{I_h}^2,\\
\|u(t)v(t)\|^2 &\le \frac{(2N+1)^2}{h}\|u(t)\|^2\|v(t)\|^2.
\end{aligned}$$

Lemma 7 *If $u(x,t)\in V_N^3$ for all $x_3\in I_h$ and $t\in S_\tau$, then*

$$\|u(t)\|_{l^4}^4\le\frac{8\pi}{h}\|u(t)\|^2\|u(t)\|_1^2.$$

Lemma 8 *If $h<2\varepsilon_1$ and $\varepsilon_1>0$, then for all $\tilde{x}_3\in\dot{I}_h$ and $t\in S_\tau$,*

$$\|u(\tilde{x}_3,t)\|_Q^2\le\varepsilon(\|u_{\bar{x}_3}(t)\|^2+\|u_{x_3}(t)\|^2)+c(\varepsilon)\|u(t)\|^2,$$

where $c(\varepsilon)$ is a positive constant depending only on $\varepsilon=\frac{\varepsilon_1^2}{2\varepsilon_1-h}$.

Lemma 9 *If $u\in[H_p^\beta(Q)]^3$ and $v\in V_N^3$, then*

$$\begin{aligned}
\|P_Nu-u\|_\alpha &\le cN^{\alpha-\beta}|u|_\beta, && 0\le\alpha\le\beta,\\
\|P_cu-u\|_\alpha &\le cN^{\alpha-\beta}|u|_\beta, && 0\le\alpha\le\beta,\beta>1\\
\|Rv-v\|_\alpha &\le cN^{\alpha-\beta}|v|_\beta, && 0\le\alpha\le\beta,r\ge\beta-\alpha.
\end{aligned}$$

Lemma 10 *If $u(x,t),v(x,t)\in V_N^3$ for all $x_3\in I_h$ and $t\in S_\tau$, then*

$$\|H(u(t),v(t))\|^2\le\frac{cN^2}{h}\|u(t)\|^2|v(t)|_2^2.$$

Proof By Lemma 6 and Lemma 9,

$$
\begin{aligned}
\|H(u(t),v(t))\|^2 &\le \frac{cN^2}{h}[\|u^{(1)}(t)\|\|\frac{\partial}{\partial x_1}(Z(v(t)))\|^2 \\
&\quad + \|u^{(2)}(t)\|\|\frac{\partial}{\partial x_2}(Z(v(t)))\|^2 + \|u^{(3)}(t)\|\|(Z(v(t)))_{\hat{x}_3}\|^2] \\
&\le \frac{CN^2}{h}\|u(t)\|^2|v(t)|_2^2.
\end{aligned}
$$

Lemma 11 *Assume that the following conditions are fulfilled*

(i) *$E(t)$ is a non-negative function on S_τ.*

(ii) *α, β, ρ and M_l are non-negative constants.*

(iii) *$F(E)$ is such a function that if $E \le M_3$, then $F(E) \le 0$.*

(iv) *for all $t \in S_\tau, E(t) \le \rho + \tau \sum_{\substack{y\in S_\tau \\ y\le t-\tau}} [M_1E(y) + M_2N^\alpha h^{-\beta}E^2(y) + F(E(y))]$.*

(v) *$E(0) \le \rho$ and $\rho e^{(M_1+M_2)T} \le \min(M_3, \frac{h^\beta}{N^\alpha})$.*
Then for all $t \in S_\tau$ and $t \le T, E(t) \le \rho e^{(M_1+M_2)t}$. In particular, if $M_2 = 0$ and $F(E) \le 0$ for all E, then for all ρ and $t, E(t) \le e^{M_1 t}$.

Lemma 1–8 and Lemma 11 can be found in [22]. Lemma 9 follows from [23] and [24].

IV. The Error Estimation for the Problem with Dirichlet Boundary Condition

The Dirichlet boundary condition corresponds to $a = 0, b = 1$ in (1.2). Suppose that $h = O(\frac{1}{N}), \tau = O(\frac{1}{N^2})$ and

$$
\begin{cases}
\varphi^{(N)} = P_c\psi = P_c x, \\
\eta^{(N)} = P_c\xi = P_c g, \quad \text{on } \Gamma_h \times S_\tau.
\end{cases}
$$

Let $\tilde{f}_1, \tilde{f}_2, \xi_0$ and $\tilde{g}$ be the errors of f_1, f_2, ξ_0 and g respectively, which induce the errors of $\eta^{(N)}$ and $\varphi^{(N)}$, denoted by $\tilde{\eta}^{(N)}$ and $\tilde{\varphi}^{(N)}$. For simplicity, assume $\tilde{\varphi}^{(N)} = 0$ on $\Gamma_h \times S_\tau$. Then the errors satisfy the following equation

$$
\begin{cases}
\tilde{\eta}_t^{(N)}(x,t) + RJ(R(\tilde{\eta}^{(N)}(x,t) + \delta\tau\tilde{\eta}_t^{(N)}(x,t)), R(\varphi^{(N)}(x,t) + \tilde{\varphi}^{(N)}(x,t))) \\
\quad + RJ(R(\eta^{(N)}(x,t) + \delta\tau\eta_t^{(N)}(x,t)), R\tilde{\varphi}^{(N)}(x,t)) \\
\quad - RH(R\tilde{\eta}^{(N)}(x,t), R(\varphi^{(N)}(x,t) + \tilde{\varphi}^{(N)}(x,t))) \\
\quad - RH(R\eta^{(N)}(x,t), R\tilde{\varphi}^{(N)}(x,t)) - \nu\Delta(\tilde{\eta}^{(N)}(x,t) + \sigma\tau\tilde{\eta}_t^{(N)}(x,t)) \\
\quad = P_c\tilde{f}_1, \\
-\Delta\tilde{\varphi}^{(N)} = \tilde{\eta}^{(N)} + P_c\tilde{f}_2, \\
\tilde{\eta}^{(N)}(x,0) = P_c\tilde{\xi}_0(x).
\end{cases}
\tag{5.1}
$$

Besides $\tilde{\eta}^{(N)} = P_N\tilde{g}$ on $\Gamma_h \times S_\tau$.

Denote

$$
\begin{aligned}
\|u(t)\|_{q,\infty} &= \max_{\substack{x\in\Omega_h \\ r_1+r_2+r_3+r_4\le q}} \left| \left(\frac{\partial^{r_1+r_2} u}{\partial x_1^{r_1}\partial x_2^{r_2}}\right)_{\underbrace{\bar{x}_3\cdots\bar{x}_3}_{r_3}\underbrace{x_3\cdots x_3}_{r_4}} \right|, \\
|||u|||_{q,\infty} &= \max_{\substack{t\in S_\tau \\ t\le T}} \|u(t)\|_{q,\infty}.
\end{aligned}
$$

By taking the inner product with $2\tilde{\eta}^{(N)}$ in the first formula of (5.1), we obtain from (2.5), (2.7), Lemma 1 and Lemma 2 that

$$
\begin{aligned}
&\|\tilde{\eta}^{(N)}(t)\|_t^2 - \tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + 2\nu|\tilde{\eta}^{(N)}(t)|_1^2 + \nu\sigma\tau(|\tilde{\eta}^{(N)}(t)|_1^2)_t \\
&\quad -\nu\sigma\tau^2|\tilde{\eta}_t^{(N)}(t)|_1^2 - 2\delta\tau(R\tilde{\eta}_t^{(N)}(t), J(R\tilde{\eta}^{(N)}(t), R\tilde{\varphi}^{(N)}(t))) \\
&\quad +2(R\tilde{\eta}^{(N)}(t), J(R(\tilde{\eta}^{(N)}(t) + \delta\tau\tilde{\eta}_t^{(N)}(t)), R\varphi^{(N)}(t))) \\
&\quad +2(R\tilde{\eta}^{(N)}(t), J(R(\eta^{(N)}(t) + \delta\tau\eta_t^{(N)}(t)), R\tilde{\varphi}^{(N)}(t))) \\
&\quad -2(R\tilde{\eta}^{(N)}(t), H(R\tilde{\eta}^{(N)}(t), R(\varphi^{(N)}(t) + \tilde{\varphi}^{(N)}(t))) \\
&\quad +H(R\eta^{(N)}(t), R\tilde{\varphi}^{(N)}(t))) + \sum_{\alpha=1}^{3} D_\alpha(t) + B_1(t) + B_2(t) \\
&= 2(\tilde{\eta}^{(N)}(t), P_c\tilde{f}_1(t)),
\end{aligned}
\tag{5.2}
$$

where

$$
\begin{aligned}
D_1(t) &= A(R\tilde{\eta}^{(N)}(t), R\tilde{\eta}^{(N)}(t), R\tilde{\varphi}^{(N)}(t)), \\
D_2(t) &= \delta\tau A(R\tilde{\eta}_t^{(N)}(t), R\tilde{\eta}^{(N)}(t), R\tilde{\varphi}^{(N)}(t)), \\
D_3(t) &= \delta\tau A(R\tilde{\eta}^{(N)}(t), R\tilde{\eta}_t^{(N)}(t), R\tilde{\varphi}^{(N)}(t)), \\
B_1(t) &= -2\nu B(\tilde{\eta}^{(N)}(t), \tilde{\eta}^{(N)}(t)), \\
B_2(t) &= -2\nu\sigma\tau B(\tilde{\eta}^{(N)}(t), \tilde{\eta}_t^{(N)}(t)).
\end{aligned}
$$

Let m be an undetermined positive constant. By taking the inner product with $m\tau\tilde{\eta}_t^{(N)}(t)$ in the first formula of (5.1), we have from (2.5), (2.7) and Lemma 2 that

$$
\begin{aligned}
&m\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + \frac{1}{2}m\nu\tau(|\tilde{\eta}^{(N)}(t)|_1^2)_t + m\nu\tau^2(\sigma - \frac{1}{2})|\tilde{\eta}_t^{(N)}(t)|_1^2 \\
&\quad +m\tau(R\tilde{\eta}_t^{(N)}(t), J(R\tilde{\eta}^{(N)}(t), R\tilde{\varphi}^{(N)}(t))) \\
&\quad +m\tau(R\tilde{\eta}_t^{(N)}(t), J(R(\tilde{\eta}^{(N)}(t) + \delta\tau\tilde{\eta}_t^{(N)}(t)), R\varphi^{(N)}(t))) \\
&\quad +J(R(\eta^{(N)}(t) + \delta\tau\eta_t^{(N)}(t)), R\tilde{\varphi}^{(N)}(t))) \\
&\quad -m\tau(R\tilde{\eta}_t^{(N)}(t), H(R\tilde{\eta}^{(N)}(t), R(\varphi^{(N)}(t) + \tilde{\varphi}^{(N)}(t)))
\end{aligned}
$$

$$+H(R\eta^{(N)}(t), R\tilde{\varphi}^{(N)}(t))) + D_4(t) + B_3(t) + B_4(t)$$
$$= m\tau(\tilde{\eta}_t^{(N)}(t), P_c\tilde{f}_1(t)), \tag{5.3}$$

where

$$\begin{aligned} D_4(t) &= \frac{1}{2}m\delta\tau^2 A(R\tilde{\eta}_t^{(N)}(t), R\tilde{\eta}_t^{(N)}(t), R\tilde{\varphi}^{(N)}(t)), \\ B_3(t) &= -m\nu\tau B(\tilde{\eta}_t^{(N)}(t), \tilde{\eta}^{(N)}(t)), \\ B_4(t) &= -m\nu\sigma\tau^2 B(\tilde{\eta}_t^{(N)}(t), \tilde{\eta}_t^{(N)}(t)). \end{aligned}$$

Let $\varepsilon > 0$ be a small constant. Putting (5.2) and (5.3) together, we have from Lemma 9 that

$$\|\tilde{\eta}^{(N)}(t)\|_t^2 + \tau(m-1-\varepsilon)\|\tilde{\eta}_t^{(N)}(t)\|^2 + 2\nu|\tilde{\eta}^{(N)}(t)|_1^2 + \nu\tau(\sigma + \frac{m}{2})(|\tilde{\eta}^{(N)}(t)|_1^2)_t$$
$$+\nu\tau^2(m\sigma - \sigma - \frac{m}{2})|\tilde{\eta}_t^{(N)}(t)|_1^2 + \sum_{\alpha=1}^{5} G_\alpha(t) + \sum_{\alpha=1}^{4} D_\alpha(t) + \sum_{\alpha=1}^{4} B_\alpha(t)$$
$$\le \|\tilde{\eta}^{(N)}(t)\|^2 + (1 + \frac{m^2\tau}{4\varepsilon})\|\tilde{f}_1\|^2, \tag{5.4}$$

where

$$\begin{aligned} G_1(t) &= (R(2\tilde{\eta}^{(N)}(t) + m\tau\tilde{\eta}_t^{(N)}(t)), J(R(\eta^{(N)}(t) + \delta\tau\eta_t^{(N)}(t)), R\tilde{\varphi}^{(N)}(t))), \\ G_2(t) &= (R(2\tilde{\eta}^{(N)}(t) + m\tau\tilde{\eta}_t^{(N)}(t)), J(R(\tilde{\eta}^{(N)}(t) + \delta\tau\tilde{\eta}_t^{(N)}(t)), R\varphi^{(N)}(t))), \\ G_3(t) &= \tau(m-2\delta)(R\tilde{\eta}_t^{(N)}(t), J(R\tilde{\eta}^{(N)}(t), R\tilde{\varphi}^{(N)}(t))), \\ G_4(t) &= -(R(2\tilde{\eta}^{(N)}(t) + m\tau\tilde{\eta}_t^{(N)}(t)), H(R\eta^{(N)}(t), R\tilde{\varphi}^{(N)}(t))), \\ G_5(t) &= -(R(2\tilde{\eta}^{(N)}(t) + m\tau\tilde{\eta}_t^{(N)}(t)), H(R\tilde{\eta}^{(N)}(t), R\tilde{\varphi}^{(N)}(t)) + H(R\tilde{\eta}^{(N)}(t)R, \varphi^{(N)}(t))). \end{aligned}$$

On the other hand, by taking the inner product with $R^2\tilde{\varphi}^{(N)}$ in the second formula of (5.1), we have from Lemma 9 and (2.8) that

$$|R\tilde{\varphi}^{(N)}(t)|_1^2 + S(R\tilde{\varphi}^{(N)}(t)) \le \frac{1}{2c}\|R\tilde{\varphi}^{(N)}(t)\|^2 + \frac{c}{2}(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2).$$

So by Lemma 4,

$$|R\tilde{\varphi}^{(N)}(t)|_1^2 + S(R\tilde{\varphi}^{(N)}(t)) \le c(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2). \tag{5.5}$$

Moreover by Lemma 5, the second expression of (5.1) reads

$$|\tilde{\varphi}^{(N)}(t)|_2^2 \le 2(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2). \tag{5.6}$$

Now, we are going to estimate $|G_\alpha(t)|, |D_\alpha(t)|$ and $|B_\alpha(t)|$. For simplicity, let $g = g_0$ for $x_3 = 0$ and $g = g_1$ for $x_3 = 1$. Define

$$\begin{aligned} \|\tilde{g}(t)\|_Q^2 &= \|\tilde{g}_0(t)\|_Q^2 + \|\tilde{g}_1(t)\|_Q^2, \ \|\tilde{g}(t)\|_{L^4(Q)}^4 = \|\tilde{g}_0(t)\|_{L^4(Q)}^4 + \|\tilde{g}_1(t)\|_{L^4(Q)}^4, \\ \|\tilde{g}_t(t)\|_Q^2 &= \|\tilde{g}_{0t}(t)\|_Q^2 + \|\tilde{g}_{1t}(t)\|_Q^2. \end{aligned}$$

Obviously (5.5) implies

$$|G_1(t)| \le \varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + c(1+(1+\frac{\tau m^2}{\varepsilon}))\||R\eta^{(N)}(t)\||_{1,\infty}^2)(\|\tilde{\eta}^{(N)}(t)\|^2 + \|^2\tilde{f}_2(t)\|^2). \quad (5.7)$$

By a computation as in [22], we obtain

$$\begin{aligned} |G_2(t)| \;\; \le \;\; & \tau(\varepsilon + \frac{c\tau\||R\varphi^{(N)}\||_{2,\infty}^2}{\varepsilon})\|\tilde{\eta}_t^{(N)}(t)\|^2 + \varepsilon\nu|\tilde{\eta}^{(N)}(t)|_1^2 \\ & + \frac{c\||R\varphi^{(N)}\||_{2,\infty}^2}{\varepsilon}\|\tilde{\eta}^{(N)}(t)\|^2 \\ & + c(1+\frac{\||R\varphi^{(N)}\||_{2,\infty}^2}{\varepsilon})(\frac{\tau}{h}\|\tilde{g}(t)\|_Q^2 + \tau h\|\tilde{g}_t(t)\|_Q^2). \end{aligned} \quad (5.8)$$

We have from Lemma 6 and (5.5) that

$$|G_3(t)| \le \varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + \frac{c\tau N^2}{\varepsilon h}(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2)|\tilde{\eta}^{(N)}(t)|_1^2. \quad (5.9)$$

Also, Lemma 7, Lemma 10 and (5.6) lead to

$$\begin{aligned} |G_4(t)| \;\; \le \;\; & \varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + \frac{c}{h}(1+\frac{\tau N^2}{\varepsilon})\|\tilde{\eta}^{(N)}(t)\|^4 + \frac{c}{h}\tilde{\eta}^{(N)}(t)\|^2|\tilde{\eta}^{(N)}(t)|_1^2 \\ & + c(1+\frac{\tau N^2}{\varepsilon h}\|\tilde{f}_2(t)\|^2)\|\tilde{\eta}^{(N)}(t)\|^2 + c\|\tilde{f}_2(t)\|^2). \end{aligned} \quad (5.10)$$

Clearly

$$\begin{aligned} |G_5(t)| \;\; \le \;\; & \varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + c(1+\frac{\tau}{\varepsilon})(\||R\eta^{(N)}\||_{0,\infty}^2 + \||R\varphi^{(N)}\||_{2,\infty}^2) \\ & \cdot (\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2). \end{aligned} \quad (5.11)$$

Next, for any $a^* > 0$,

$$\begin{aligned} & |(u(0,t), v(h,t))_Q + (v(1-h,t), u(1,t))_Q| \\ & \quad \le \frac{h}{2a^*}(\|v(0,t)\|_Q^2 + \|v(1,t)\|_Q^2) + \frac{a^*}{2h}(\|u(h,t)\|_Q^2 + \|u(1-h,t)\|_Q^2). \end{aligned}$$

Thus by Lemma 6, Lemma 8, Lemma 9 and (5.6),

$$\begin{aligned} |D_1(t)| \le & \varepsilon\nu S(\tilde{\eta}^{(N)}(t)) + \frac{chN^2}{\varepsilon}\nu\|\tilde{g}(t)\|_Q^2(\|Z^{(3)}(R\tilde{\varphi}^{(N)}(h,t))\|_Q^2 \\ & +\|Z^{(3)}(R\tilde{\varphi}^{(N)}(1-h,t))\|_Q^2) \\ \le & \varepsilon\nu S(\tilde{\eta}^{(N)}(t)) + \frac{chN^2}{\varepsilon\nu}\|\tilde{g}(t)\|_Q^2(|R\tilde{\varphi}^{(N)}(t)|_2^2 + |R\tilde{\varphi}^{(N)}(t))|_1^2) \\ \le & \varepsilon\nu S(\tilde{\eta}^{(N)}(t)) + \frac{chN^2}{\varepsilon\nu}\|\tilde{g}(t)\|_Q^2(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t))\|^2). \end{aligned} \quad (5.12)$$

Similarly,

$$|D_2(t)| + |D_3(t)| \le \varepsilon\nu S(\tilde{\eta}^{(N)}(t)) + \varepsilon\nu\tau^2 S(\tilde{\eta}_t^{(N)}(t))$$
$$+\frac{chN^2}{\varepsilon\nu}(\|\tilde{g}(t)\|_Q^2 + \tau^2\|\tilde{g}_t(t)\|_Q^2)(\|\tilde{\eta}^{(N)}(t))\|^2 + \|\tilde{f}_2(t)\|^2), \quad (5.13)$$

$$|D_4(t)| \le \varepsilon\nu\tau^2 S(\tilde{\eta}_t^{(N)}(t)) + \frac{ch\tau^2 N^2}{\varepsilon\nu}\|\tilde{g}_t(t)\|_Q^2(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2). \quad (5.14)$$

Finally (see [22]),

$$B_1(t) \ge 2\nu S(\tilde{\eta}^{(N)}(t)) - \frac{c}{\varepsilon h}\|\tilde{g}(t)\|_Q^2, \quad (5.15)$$

$$B_2(t) + B_3(t) \ge \nu\tau(\sigma + \frac{m}{2})[S(\tilde{\eta}^{(N)}(t))]_t - \nu\tau^2(\sigma + \frac{m}{2})S(\tilde{\eta}^{(N)}(t))$$
$$-\varepsilon\nu S(\tilde{\eta}^{(N)}(t)) - \varepsilon\nu\tau^2 S(\tilde{\eta}_t^{(N)}(t)) - \frac{c}{\varepsilon h}(\|\tilde{g}(t)\|_Q^2 + \tau h^2\|\tilde{g}_t(t)\|_Q^2), \quad (5.16)$$

$$B_4(t) \ge m\nu\sigma\tau^2 S(\tilde{\eta}_t^{(N)}(t)) - \frac{c\tau h}{\varepsilon}\|\tilde{g}_t(t)\|_Q^2. \quad (5.17)$$

We substitute (5.7)–(5.17) into (5.4) and then obtain

$$\|\tilde{\eta}^{(N)}(t)\|_t^2 + \tau(m - 1 - 6\varepsilon - \frac{c\tau}{\varepsilon}|||R\varphi^{(N)}|||_{2,\infty}^2)\|\tilde{\eta}_t^{(N)}(t)\|^2 + \nu|\tilde{\eta}^{(N)}(t)|_1^2$$
$$+\nu\tau(\sigma + \frac{m}{2})(|\tilde{\eta}^{(N)}(t)|_1^2)_t + \nu\tau^2(m\sigma - \sigma - \frac{m}{2})|\tilde{\eta}_t^{(N)}(t)|_1^2$$
$$+\nu(2 - 3\varepsilon)S(\tilde{\eta}^{(N)}(t)) + \nu\tau(\sigma + \frac{m}{2})[S(\tilde{\eta}^{(N)}(t))]_t$$
$$+\nu\tau^2(m\sigma - \sigma - \frac{m}{2} - 3\varepsilon)S(\tilde{\eta}_t^{(N)}(t))$$
$$\le F_0(t)\|\tilde{\eta}^{(N)}(t)\|^2 + F_1(t)\|\tilde{\eta}^{(N)}(t)\|^4 + F_2(t)|\tilde{\eta}^{(N)}(t)|_1^2 + R(t), \quad (5.18)$$

where

$$\begin{aligned}
F_0(t) &= c + \frac{c}{\varepsilon}(\tau + \varepsilon)(|||R\eta^{(N)}|||_{1,\infty}^2 + |||R\varphi^{(N)}|||_{2,\infty}^2) \\
&\quad + \frac{c}{\varepsilon h}\|\tilde{f}_2(t)\|^2 + \frac{chN^2}{\varepsilon\nu}(\|\tilde{g}(t)\|_Q^2 + \tau^2\|\tilde{g}_t(t)\|_Q^2), \\
F_1(t) &= \frac{c}{\varepsilon h}(1 + \varepsilon), \\
F_2(t) &= -\nu - \varepsilon\nu + \frac{c}{\varepsilon h}(1 + \varepsilon)\|\tilde{\eta}^{(N)}(t)\|^2 + \frac{c}{\varepsilon h}\|\tilde{f}_2(t)\|^2, \\
\tilde{R}(t) &= \frac{c}{\varepsilon}(\tau + \varepsilon)\|\tilde{f}_1(t)\|^2 + \frac{c}{\varepsilon}[\varepsilon + (\tau + \varepsilon)(|||R\eta^{(N)}|||_{1,\infty}^2 + |||R\varphi^{(N)}|||_{2,\infty}^2) \\
&\quad + \frac{hN^2}{\varepsilon\nu}(\|\tilde{g}(t)\|_Q^2 + \tau^2\|\tilde{g}_t(t)\|_Q^2)]\|\tilde{f}_2(t)\|^2 \\
&\quad + \frac{c}{\varepsilon h}(|||R\varphi^{(N)}|||_{2,\infty}^2 + 1 + \varepsilon)(\|\tilde{g}(t)\|_Q^2 + \tau^2\|\tilde{g}_t(t)\|_Q^2).
\end{aligned}$$

Now, let ε be suitably small and $\tau < \frac{\varepsilon^2}{c|||R\varphi^{(N)}|||^2_{2,\infty}}$. Then we choose the value of m as follows:

Case I, $\sigma > \frac{1}{2}$. We take

$$m > m_1 = \max\left(\frac{2\sigma + 6\varepsilon}{2\sigma - 1}, 1 + P_0 + 7\varepsilon\right), \quad P_0 \geq 0.$$

Then (5.18) becomes

$$\begin{aligned}
&(\|\tilde{\eta}^{(N)}(t)\|^2)_t + P_0\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + \nu[|\tilde{\eta}^{(N)}(t)|_1^2 + S(\tilde{\eta}^{(N)}(t))] \\
&\quad + \nu\tau(\sigma + \frac{m}{2})[|\tilde{\eta}^{(N)}(t)|_1^2 + S(\tilde{\eta}^{(N)}(t))]_t \\
&\quad \leq F_0(t)\|\tilde{\eta}^{(N)}(t)\|^2 + F_1(t)\|\tilde{\eta}^{(N)}(t)\|^4 + F_2(t)|\tilde{\eta}^{(N)}(t)|_1^2 + \tilde{R}(t).
\end{aligned} \tag{5.19}$$

Case II, $\sigma > \frac{1}{2}$. By Lemma 3,

$$\begin{aligned}
|\tilde{\eta}_t^{(N)}(t)|_1^2 &\leq (2N^2 + \frac{4}{h^2})\|\tilde{\eta}_t^{(N)}(t)\|^2 + \frac{2}{h}\|\tilde{g}_t(t)\|_Q^2, \\
S(\tilde{\eta}_t^{(N)}(t)) &\leq \frac{1}{2h^2}\|\tilde{\eta}_t^{(N)}(t)\|^2.
\end{aligned}$$

Thus

$$\begin{aligned}
&\tau(m - 1 - 7\varepsilon)\|\tilde{\eta}_t^{(N)}(t)\|^2 - \nu\tau^2(\frac{1}{2} + 3\varepsilon)(|\tilde{\eta}_t^{(N)}(t)|_1^2 + S(\tilde{\eta}_t^{(N)}(t))) \\
&\quad \geq \tau(m - 1 - 7\varepsilon - \nu\tau(1 + 6\varepsilon)(N^2 + \frac{9}{4h^2}))\|\tilde{\eta}_t^{(N)}(t)\|^2 - \frac{c\tau h}{\varepsilon}\|\tilde{g}_t(t)\|_Q^2.
\end{aligned}$$

We choose

$$m > m_2 = 1 + P_0 + \nu\tau(1 + 6\varepsilon)(N^2 + \frac{9}{4h^2}) + 7\varepsilon.$$

Then (5.19) still holds.

Case III, $\sigma < \frac{1}{2}$ and $\tau < \frac{4h^2}{\nu(1-2\sigma)(9+4N^2h^2)}$. We take

$$m > m_3 = [1 + P_0 + 2\nu\tau(\sigma + 3\varepsilon)(N^2 + \frac{9}{4h^2}) + 7\varepsilon][1 - \nu\tau(1 - 2\sigma)(N^2 + \frac{9}{4h^2})]^{-1}.$$

Then (5.19) still holds.

Now set

$$\begin{aligned}
E_1^{(N)}(t) &= \|\tilde{\eta}^{(N)}(t)\|^2 + \nu\tau(|\tilde{\eta}^{(N)}(t)|_1^2 + S(\tilde{\eta}^{(N)}(t))) \\
&\quad + \tau\sum_{\substack{y\in S_\tau \\ y\leq t-\tau}}(P_0\tau\|\tilde{\eta}_t^{(N)}(y)\|^2 + \nu|\tilde{\eta}^{(N)}(y)|_1^2 + \nu S(\tilde{\eta}^{(N)}(y))) \\
\rho_1^{(N)}(t) &= \|\tilde{\eta}^{(N)}(0)\|^2 + \tau\sum_{\substack{y\in S_\tau \\ y\leq t-\tau}}\tilde{R}(y).
\end{aligned}$$

By summing up (5.19) for $t \in S_\tau$, we obtain

$$E_1^{(N)}(t) \le \tilde{\rho}_1^{(N)}(t) + \tau \sum_{\substack{y\in S_\tau \\ y \le t-\tau}} (F_0(y)E_1^{(N)}(y) + F_1(y)(E_1^{(N)}(y))^2 + F_2(y)|\tilde{\eta}^{(N)}(y)|_1^2).$$

Finally by using Lemma 11, we get the following theorem.

Theorem 1 *If the following conditions are fulfilled*

(i) $h = O(\frac{1}{N})$ *and* $\tau = O(\frac{1}{N^2})$,

(ii) $\sigma \ge \frac{1}{2}$ *or* $\tau < \frac{4h^2}{\nu(1-2\sigma)(9+4N^2h^2)}$,

(iii) for all $t \le T, \|\tilde{f}_2(t)\|^2 \le \frac{b_1}{N}, \|\tilde{g}(t)\|_Q^2 \le \frac{b_2}{N}, \|\tilde{\rho}_1^{(N)}(t)\|^2 \le \frac{b_3}{N}$,

then for all $t \le T, E_1^{(N)}(t) \le b_4 e^{b_5 t}\rho_1^{(N)}(t)$, *where* b_α *are suitably small positive constants depending only on* $|||R\eta^{(N)}|||_{1,\infty}^2, |||R\varphi^{(N)}|||_{2,\infty}^2$ *and* ν.

Remark 1 Because we adopt the skew symmetric decomposition, so the main errors

$$(RZ^{(N)}(t), J(RZ^{(N)}(t), R\tilde{\varphi}^{(N)}(t))), \quad Z = \tilde{\eta}^{(N)} \text{ or } \tau\tilde{\eta}_t^{(N)}$$

depend only on the boundary errors and thus Theorem 1 follows. Otherwise we require that $\rho_1^{(N)}(t) \le \frac{b_3 h}{N^2}$.

We now turn to the convergence. For simplicity of expressions, for non- negative integers m and n, let

$$H^m(H^n(Q), I) = \{u | \frac{\partial^{p+q} u}{\partial x_1^r \partial x_2^{p-r} \partial x_3^q} \in L^2(\Omega),\ 0 \le r \le p, 0 \le p \le n, 0 \le q \le m\}.$$

For any real numbers α and β, $H^\alpha(H^\beta(Q), I)$ is defined by the interpolation of spaces with the norm $\|\cdot\|_{H^\alpha(H^\beta(Q),I)}$. Similarly we can define the space $C^m(H^\alpha(Q), I)$. Furthermore

$$C^m(0,T; H^\alpha(\Omega)) = \{u | \frac{\partial^p u}{\partial t^p} \in H^\alpha(\Omega) \text{ is continuous in } t \in (0,T], 0 \le p \le m\}$$

equipped with the norm $|||\cdot|||_{H^\alpha(\Omega)}$. Similarly, we can define the space $C^m(0,T; H^\alpha(H^\beta(Q,I))$ with the norm $|||\cdot|||_{H^\alpha(H^\beta(Q),I)}$.

Let $\xi^{(N)} = P_N\xi$ and $\psi^{(N)} = P_N\psi$. By (1.1),

$$\begin{cases} \xi_t^{(N)} + RJ(R(\xi^{(N)} + \delta\tau\xi_t^{(N)}), R\psi^{(N)}) - RH(R\xi^{(N)}, R\psi^{(N)}) \\ -\nu\Delta(\xi^{(N)} + \sigma\tau\xi_t^{(N)}) = P_N f_1 + \sum_{\alpha=1}^{7} M_\alpha, \\ -\Delta\psi^{(N)} = \xi^{(N)} + P_N f_2 + M_7, \\ \xi^{(N)}(0) = P_N\xi_0, \end{cases} \tag{5.20}$$

where

$$M_1(t) = \xi_t^{(N)} - \frac{\partial \xi^{(N)}}{\partial t}, \qquad M_2(t) = RJ(R\xi^{(N)}, R\psi^{(N)}) - P_N[(\nabla \times \psi)\cdot \nabla]\xi,$$
$$M_3(t) = \delta\tau RJ(R\xi_t^{(N)}, R\psi^{(N)}), \quad M_4(t) = P_N[(\xi\cdot\nabla)(\nabla\times\psi)] - RH(R\xi^{(N)}, R\psi^{(N)}),$$
$$M_5(t) = \nu\frac{\partial^2 \xi^{(N)}}{\partial x_3^2} - \nu\xi^{(N)}_{x_3\bar{x}_3}, \qquad M_6(t) = \nu\sigma\tau\Delta\xi_t^{(N)}, \quad M_7(t) = \frac{\partial^2\psi^{(N)}}{\partial x_3^2} - \psi^{(N)}_{x_3\bar{x}_3}.$$

Let $\eta = \xi^{(N)} + \tilde{\xi}$ and $\varphi = \psi^{(N)} + \tilde{\psi}$. Then we have from (2.2) and (5.20) that

$$\begin{cases} \tilde{\xi}_t^{(N)} + RJ(R(\tilde{\xi}^{(N)} + \delta\tau\tilde{\xi}_t^{(N)}), R(\psi^{(N)} + \tilde{\psi}^{(N)})) \\ \quad + RJ(R(\xi^{(N)} + \delta\tau\xi_t^{(N)}), R\tilde{\psi}^{(N)}) - \nu\Delta(\tilde{\xi}^{(N)} + \sigma\tau\tilde{\xi}_t^{(N)}) \\ \quad - RH(R\tilde{\xi}^{(N)}, R(\psi^{(N)} + \tilde{\psi}^{(N)})) - RH(P\xi^{(N)}, R\tilde{\psi}^{(N)}) \\ \quad = P_c f_1 - P_N f_1 - \sum_{\alpha=1}^{6} M_\alpha(t), \\ -\Delta\tilde{\psi}^{(N)} = \tilde{\xi}^{(N)} + P_c f_2 - P_N f_2 - M_7(t), \\ \tilde{\xi}^{(N)}(0) = P_c\xi_0 - P_N\xi_0. \end{cases} \tag{5.21}$$

Besides, $\tilde{\xi}^{(N)} = P_c g - P_N g$ on $\Gamma_h \times S_\tau$.

Let $\beta > 1$ and $\mu > 0$. We have from Lemma 9 and embedding theory that

$$\begin{aligned} \|M_1(t)\| &\le c\tau|||\frac{\partial^2\xi}{\partial t^2}|||_{H^{\frac{1}{2}+\mu}(L^2(Q),I)}, \\ \|M_2(t)\| &\le c(N^{-\beta} + h^2)[|||\psi|||_{H^{\frac{5}{2}+\mu}(\Omega)}(|||\xi|||_{H^{\frac{1}{2}+\mu}(H^{\beta+1}(Q),I)} + |||\xi|||_{H^{\frac{3}{2}+\mu}(H^\beta(Q),I)} \\ &\quad + |||\xi|||_{H^{\frac{7}{2}+\mu}(L^2(Q),I)}) + |||\xi|||_{H^{\frac{5}{2}+\mu}(\Omega)}(|||\psi|||_{H^{\frac{1}{2}+\mu}(H^{\beta+1}(Q),I)} \\ &\quad + |||\psi|||_{H^{\frac{3}{2}+\mu}(H^\beta(Q),I)} + |||\psi|||_{H^{\frac{7}{2}+\mu}(L^2(Q),I)})], \\ \|M_3(t)\| &\le c\tau|||\psi|||_{H^{\frac{5}{2}+\mu}(\Omega)}(|||\frac{\partial\xi}{\partial t}|||_{H^{\frac{1}{2}+\mu}(H^1(Q),I)} + |||\frac{\partial\xi}{\partial t}|||_{H^{\frac{3}{2}+\mu}(L^2(Q),I)}), \\ \|M_4(t)\| &\le c(N^{-\beta} + h^2)[|||\xi|||_{H^{\frac{3}{2}+\mu}(\Omega)}(|||\psi|||_{H^{\frac{1}{2}+\mu}(H^{\beta+2}(Q),I)} + |||\psi|||_{H^{\frac{3}{2}+\mu}(H^{\beta+1}(Q),I)} \\ &\quad + |||\psi|||_{H^{\frac{5}{2}+\mu}(H^\beta(Q),I)}) + |||\psi|||_{H^{\frac{7}{2}+\mu}(H^1(Q),I)} + |||\psi|||_{H^{\frac{9}{2}+\mu}(L^2(Q),I)}) \\ &\quad + |||\xi|||_{H^{\frac{1}{2}+\mu}(H^\beta(Q),I)}|||\psi|||_{H^{\frac{7}{2}+\mu}(\Omega)}], \\ \|M_5\| &\le ch^2|||\xi|||_{H^{\frac{9}{2}+\mu}(L^2(Q),I)}, \\ \|M_6(t)\| &\le c\tau(|||\frac{\partial\xi}{\partial t}|||_{H^{\frac{5}{2}+\mu}(L^2(Q),I)} + |||\frac{\partial\xi}{\partial t}|||_{H^{\frac{1}{2}+\mu}(H^2(Q),I)}), \\ \|M_7\| &\le ch^2|||\psi|||_{H^{\frac{9}{2}+\mu}(L^2(Q),I)}, \end{aligned}$$

and

$$\|P_c Z - P_N Z\| \le cN^{-\beta}\|Z\|_{H^{\frac{1}{2}+\mu}(H^\beta(Q),I)}, \quad \|P_c Z - P_N Z\|_q \le cN^{-\beta}\|Z\|_{H^\beta(Q)}.$$

Finally, by an argument as in the proof of Theorem 1, we have the following conclusion.

Theorem 2 *Assume that*

(i) The conditions (i) and (ii) of Theorem 1 hold,

(ii) For $\beta > 1$ and $\mu > 0$,

$$\xi \in C(0,T; H^{\frac{5}{2}+\mu}(\Omega) \bigcap H^{\frac{1}{2}+\mu}(H^{\beta+1}(Q), I) \bigcap H^{\frac{3}{2}+\mu}(H^{\beta}(Q), I) \bigcap H^{\frac{9}{2}+\mu}(L^2(Q), I)),$$
$$\frac{\partial \xi}{\partial t} \in C(0,T; H^{\frac{5}{2}+\mu}(L^2(Q), I) \bigcap H^{\frac{1}{2}+\mu}(H^2(Q), I)),$$
$$\frac{\partial^2 \xi}{\partial t^2} \in C(0,T; H^{\frac{1}{2}+\mu}(L^2(Q), I)),$$
$$\psi \in C(0,T; H^{\frac{7}{2}+\mu}(\Omega) \bigcap H^{\frac{1}{2}+\mu}(H^{\beta+2}(Q), I) \bigcap H^{\frac{3}{2}+\mu}(H^{\beta+1}(Q), I)$$
$$\bigcap H^{\frac{5}{2}+\mu}(H^{\beta}(Q), I) \cap H^{\frac{7}{2}+\mu}(H^1(Q), I) \bigcap H^{\frac{9}{2}+\mu}(L^2(Q), I)),$$

(iii) $f_1, f_2 \in C(0,T; H^{\frac{1}{2}+\mu}(H^{\beta}(Q), I))$, $\quad g \in C(0,T; H^{\beta+\frac{1}{2}}(Q))$.

Then for all $t \le T$, $\|\xi(t) - \eta^{(N)}(t)\|^2 \le b_1^*(\tau^2 + h^4 + N^{-2\beta})$, where b_1^* is a positive constant depending only on ν and the norms of ξ, ψ, f_1, f_2 and g in the spaces mentioned in the above.

Remark 2 If we take $\eta^{(N)}(x,t) = P_N g(x,t)$ for $x_3 = 0, 1$, then we do not require that $g \in C(0,T; H^{\beta+\frac{1}{2}}(Q))$.

VI. The Error Estimations for other Problems

In this section, we discuss the case where $a > 0$. The corresponding boundary approximation is the following

$$a\eta_n^{(N)}(x,t) + b\bar{\eta}^{(N)}(x,t) = P_c g(x,t), \quad (x,t) \in \Gamma_h \times S_\tau, \tag{6.1}$$

where

$$\bar{\eta}^{(N)}(x,t) = \begin{cases} \frac{1}{2}(\eta^{(N)}(x + he_3, t) + \eta^{(N)}(x,t)), & \text{for } x_3 = 0, \\ \frac{1}{2}(\eta^{(N)}(x - he_3, t) + \eta^{(N)}(x,t)), & \text{for } x_3 = 1 \end{cases}$$

and

$$\eta_n^{(N)}(x,t) = \begin{cases} -\eta_{x_3}^{(N)}(x,t), & \text{for } x_3 = 0, \\ \eta_{\bar{x}_3}^{(N)}(x,t), & \text{for } x_3 = 1. \end{cases}$$

For simplicity, let $\delta = 0$ and $\tilde{\varphi}^{(N)} = 0$ on $\Gamma_h \times S_\tau$. By an argument as in Section V, we obtain

$$\|\tilde{\eta}^{(N)}(t)\|_t^2 + \tau(m - 1 - \varepsilon)\|\tilde{\eta}_t^{(N)}(t)\|^2 + 2\nu|\tilde{\eta}^{(N)}(t)|_1^2 + \nu\tau(\sigma + \frac{m}{2})(|\tilde{\eta}^{(N)}(t)|_1^2)_t$$
$$+\nu\tau^2(m\sigma - \sigma - \frac{m}{2})|\tilde{\eta}_t^{(N)}(t)|_1^2 + D_1(t) + \sum_{\alpha=4}^{7} G_\alpha(t) + \sum_{\alpha=1}^{4} B_\alpha(t)$$
$$\le \|\tilde{\eta}^{(N)}(t)\|^2 + c(1 + \frac{m^2\tau}{4\varepsilon})\|\tilde{f}_1(t)\|^2, \tag{6.2}$$

where $D_1(t), G_4(t), G_5(t)$ and $B_\alpha(t)$ are the same as in Section V,

$$\begin{aligned}
G_6(t) &= (2\tilde{\eta}^{(N)}(t) + m\tau\tilde{\eta}_t^{(N)}(t), RJ(R\eta^{(N)}(t), R\tilde{\varphi}^{(N)}(t)) + RJ(R\tilde{\eta}^{(N)}(t), R\varphi^{(N)}(t))),\\
G_7(t) &= m\tau(\tilde{\eta}_t^{(N)}(t), RJ(R\tilde{\eta}^{(N)}(t), R\tilde{\varphi}^{(N)}(t)).
\end{aligned}$$

We first consider the error estimation for $b=0$. By Lemma 7, Lemma 8, and Lemma 9, we obtain

$$\begin{aligned}
|D_1(t)| \le {} & \frac{\varepsilon}{32\pi}(\|\tilde{\eta}^{(N)}(0,t)\|^4_{L^4(Q)} + \|\tilde{\eta}^{(N)}(h,t)\|^4_{L^4(Q)} + \|\tilde{\eta}^{(N)}(1-h,t)\|^4_{L^4(Q)}\\
& + \|\tilde{\eta}^{(N)}(1,t)\|^4_{L^4(Q)}) + \frac{c}{\varepsilon}(\|Z^{(3)}(\tilde{\varphi}^{(N)}(1-h,t))\|^2_Q + \|Z^{(3)}(\tilde{\varphi}^{(N)}(h,t))\|^2_Q)\\
\le {} & \frac{\varepsilon}{8\pi h}\|\tilde{\eta}^{(N)}(t)\|^4_{l^4} + \frac{c}{\varepsilon}(|\tilde{\varphi}^{(N)}(t)|^2_2 + |\tilde{\varphi}^{(N)}(t)|^2_1) + c\varepsilon h^4\|\tilde{g}(t)\|^4_{L^4(Q)}\\
\le {} & \frac{\varepsilon}{h^2}\|\tilde{\eta}^{(N)}(t)\|^2\|\tilde{\eta}^{(N)}(t)\|^2_1 + \frac{c}{\varepsilon}(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2)\\
& + c\varepsilon h^4\|\tilde{g}(t)\|^4_{L^4(Q)}. \qquad (6.3)
\end{aligned}$$

Also, we have

$$\begin{aligned}
|G_6(t)| &\le \varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + \varepsilon\nu|\tilde{\eta}^{(N)}(t)|^2_1 + \frac{c}{\varepsilon}(|||R\eta^{(N)}|||^2_{1,\infty} + |||R\varphi^{(N)}|||^2_{2,\infty})\\
&\quad \cdot(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2 + \tau h\|\tilde{g}(t)\|^2_Q),\\
|G_7(t)| &\le \varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + \frac{c\tau N^2}{\varepsilon h}|\tilde{\eta}^{(N)}(t)|^2_1(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2),\\
|B_1(t)| &\le \varepsilon\nu|\tilde{\eta}^{(N)}(t)|^2_1 + \frac{c}{\varepsilon}(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{g}(t)\|^2_Q),\\
|B_2(t)| &\le \varepsilon\nu|\tilde{\eta}^{(N)}(t)|^2_1 + \frac{c}{\varepsilon}(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{g}(t)\|^2_Q + \tau^2\|\tilde{g}_t(t)\|^2_Q),\\
|B_3(t)| &\le \varepsilon\nu\tau^2|\tilde{\eta}_t^{(N)}(t)|^2_1 + \frac{c\tau^2}{\varepsilon}\|\tilde{\eta}_t^{(N)}(t)\|^2 + \frac{c}{\varepsilon}(\|\tilde{g}(t)\|^2_Q + \tau^2\|\tilde{g}_t(t)\|^2_Q),\\
|B_4(t)| &\le \varepsilon\nu\tau^2|\tilde{\eta}_t^{(N)}(t)|^2_1 + \frac{c\tau^2}{\varepsilon}(\|\tilde{\eta}_t^{(N)}(t)\|^2 + \|\tilde{g}_t(t)\|^2_Q).
\end{aligned}$$

By an argument as in the proof of Theorem 1, we obtain the following result.

Theorem 3 *Assume that*

(i) $a=1, b=0, \delta=0, h=O(\frac{1}{N}), \tau=O(\frac{1}{N^2})$,

(ii) $\sigma \ge \frac{1}{2}$ *or* $\tau < \dfrac{h^2}{\nu(1-2\sigma)(2+N^2h^2)}$,

(iii) for all $t \le T, \|\tilde{f}_2(t)\|^2 \le \dfrac{b_6}{N}, \rho_2^{(N)}(t) \le \dfrac{b_7}{N^2}$.

Then for all $t \le T$,

$$E_2^{(N)}(t) \le b_8 e^{b_9 t} \rho_2^{(N)}(t),$$

where

$$\begin{aligned} E_2^{(N)}(t) &= \|\tilde{\eta}^{(N)}(t)\|^2 + \nu\tau|\tilde{\eta}^{(N)}(t)|_1^2 + \tau \sum_{\substack{y\in S_\tau \\ y\in t-\tau}} (p_0\tau\|\tilde{\eta}_t^{(N)}(y)\|^2 + \nu|\tilde{\eta}^{(N)}(y)|_1^2), \\ \rho_2^{(N)}(t) &= \|\tilde{\eta}^{(N)}(0)\|^2 + \sum_{\substack{y\in S_\tau \\ y\in t-\tau}} (\|\tilde{f}_1(y)\|^2 + \|\tilde{f}_2(y)\|^2 \\ &\quad + \|\tilde{g}(y)\|_Q^2 + \tau^2\|\tilde{g}_t(t)\|_Q^2 + h^4\|\tilde{g}(t)\|_{L^4(Q)}^2). \end{aligned}$$

As to the case with $a = 1$ and $b > 0$, let

$$S^*(\tilde{\eta}^{(N)}(t)) = \frac{1}{2}(\|\tilde{\eta}^{(N)}(0,t) + \tilde{\eta}^{(N)}(h,t)\|_Q^2 + \|\tilde{\eta}^{(N)}(1-h,t) + \tilde{\eta}^{(N)}(1,t)\|_Q^2).$$

By (6.1),

$$\begin{cases} -\tilde{\eta}_{x_3}^{(N)}(x_1,x_2,0,t) = P_c\tilde{g}_0(t) - \dfrac{b}{2}(\tilde{\eta}^{(N)}(x_1,x_2,0,t) + \tilde{\eta}^{(N)}(x_1,x_2,h,t)), \\ \tilde{\eta}_{\bar{x}_3}^{(N)}(x_1,x_2,1,t) = P_c\tilde{g}_1(t) - \dfrac{b}{2}(\tilde{\eta}^{(N)}(x_1,x_2,1,t) + \tilde{\eta}^{(N)}(x_1,x_2,1-h,t)). \end{cases}$$

Thus (6.3) is valid also. Moreover

$$\|\tilde{\eta}_{x_3}^{(N)}(0,t)\|_Q^2 + \|\tilde{\eta}_{\bar{x}_3}^{(N)}(1,t)\|_Q^2 \le \varepsilon S^*(\tilde{\eta}^{(N)}(t)) + \frac{c}{\varepsilon}\|\tilde{g}(t)\|_Q^2.$$

Therefore

$$\begin{aligned} |G_6(t)| + |G_7(t)| \le\ & \varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + \varepsilon\nu|\tilde{\eta}^{(N)}(t)|_1^2 \\ & + \frac{c}{\varepsilon}(|||R\eta^{(N)}|||_{1,\infty}^2 + |||R\varphi^{(N)}|||_{2,\infty}^2) \\ & (\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2 + \tau h S^*(\tilde{\eta}^{(N)}(t)) + \tau h\|\tilde{g}(t)\|_Q^2) \\ & + \frac{c\tau N^2}{\varepsilon h^2}|\tilde{\eta}^{(N)}(t)|_1^2(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2). \end{aligned}$$

On the other hand (see [22]),

$$\begin{aligned} B_1(t) &\ge b\nu(1-\varepsilon)S^*(\tilde{\eta}^{(N)}(t)) - \frac{c}{\varepsilon}\|\tilde{g}(t)\|_Q^2, \\ B_2(t) + B_3(t) &\ge \frac{1}{4}b\nu\tau(2\sigma+m)S_t^*(\tilde{\eta}^{(N)}(t)) - \frac{1}{4}b\nu\tau^2(2\sigma+m+\varepsilon)S^*(\tilde{\eta}_t^{(N)}(t)) \\ &\quad - \varepsilon b\nu S^*(\tilde{\eta}^{(N)}(t)) - \frac{c}{\varepsilon}(\|\tilde{g}(t)\|_Q^2 + \tau^2\|\tilde{g}_t(t)\|_Q^2), \\ B_4(t) &\ge \frac{1}{2}b\nu\tau^2(m\sigma - \varepsilon)S^*(\tilde{\eta}^{(N)}(t)) - \frac{c\tau^2}{\varepsilon}\|\tilde{g}_t(t)\|_Q^2. \end{aligned}$$

By an argument as in the proof of Theorem 1, we get the following result.

Theorem 4 *Assume that*

(i) $a = 1, b > 0, h = O(\frac{1}{N}), \tau = O(\frac{1}{N^2})$,

(ii) $\sigma \geq \frac{1}{2}$ *or* $\tau < \frac{2h^2}{\nu(1-2\sigma)(4+2N^2h^2+b)}$,

(iii) for all $t \leq T, \|\tilde{f}_2(t)\|^2 \leq \frac{b_{10}}{N}, \rho_3^{(N)}(t) \leq \frac{b_{11}}{N^2}$.

Then for all $t \leq T$,

$$E_3^{(N)}(t) \leq b_{12} e^{b_{13} t} \rho_3^{(N)}(t)$$

where $\rho_3^{(N)}$ *is the same as* $\rho_2^{(N)}(t)$*, and*

$$\begin{aligned} E_3^{(N)}(t) &= \|\tilde{\eta}^{(N)}(t)\|^2 + \nu\tau(|\tilde{\eta}^{(N)}(t)|_1^2 + S^*(\tilde{\eta}^{(N)}(t))) \\ &+ \tau \sum_{\substack{y \in S_\tau \\ y \in t-\tau}} (p_0 \tau \|\tilde{\eta}_t^{(N)}(y)\|^2 + \nu|\tilde{\eta}^{(N)}(y)|_1^2 + \nu S^*(\tilde{\eta}^{(N)}(y))). \end{aligned}$$

Remark 3 If $h^2\|\tilde{g}(t)\|_Q^2 \leq b_{16}$, then we can omit the term $h^4\|\tilde{g}(t)\|_{L^4(Q)}^4$ in $\rho_2^{(N)}(t)$ and $\rho_3^{(N)}(t)$.

We can deal with the convergence as in Section V and get the following result.

Theorem 5 *Let conditions (i), (ii) of Theorem 3 or Theorem 4 hold. In addition, condition (ii) of Theorem 2 holds and* $g \in C(0,T;H^\beta(Q))$*. Then for all* $t \leq T$ *and* $t \in S_\tau, \|\xi(t) - \eta^{(N)}(t)\|^2 \leq b_2^*(\tau^2 + h^2 + N^{-2\beta})$*, where* b_2^* *is a positive constant similar to* b_1^*.

Remark 4 If $P_c g$ is replaced by $P_N g$ in (6.1), then we do not need that $g \in C(0,T;H^\beta(Q))$.

Remark 5 If we let $I_h = \{x | x = jh - \frac{h}{2}, 1 \leq j \leq M\}$ and use the boundary approximation as before, then

$$\|\xi(t) - \eta^{(N)}(t)\|^2 \leq b_3^*(\tau^2 + h^4 + N^{-2\beta}).$$

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