If $xR \neq \{0\}$, then R has a non-zero nilpotent ideal I. That is $I^n = \{0\}$, where $n > 1$ is a fixed integer. Since $L \subseteq I$, $L^n = \{0\}$. This is a contradiction. Thus $xR = \{0\}$, so $xe = 0$ and $x \in K$. This proves the right annihilator of e in R is $\{0\}$ which is an ideal of R. By Theorem 4, R is a division ring. □

Theorem 5 *Let N be a zero-symmetric near-ring and $E = \{e \in N | en = n \ \text{ for all } \ n \in N\}$. If E is nonempty and for each $n \in N, n \neq 0$, and there is at least one element n' in N such that $nn' \in E$, then N is a near-field.*

Proof It is easily seen that $0 \notin E$ except $N = \{0\}$. If there are two different elements e, e' in E, there is at least one element n in N such that $(e - e')n \in E$, and $er = r, e'r = r$ for all $r \in N$. Hence $(e - e')r = 0$ for all $r \in N$, a contradiction. Thus $e = e'$. Let $n \in N$ and $n \neq 0$. There is at least one element m in N such that $nm = e \in E$. Hence $n(mn) = (nm)n = en = n$. There is at least one element k in N such that $(mn)k = e$. From $n(mn) = n$, we have $m[n(mn)] = (mn)(mn) = mn$. Multiplying the last equality from the right by k, we have $(mn - e)e = 0$. If $mn - e \neq 0$, there is at least one element h in N such that $(mn - e)h = e$. Multiplying $(mn - e)e = 0$ from the right by h, we have $e = 0$, a contradiction. Thus $mn = e$. Therefore $N \setminus \{0\}$ is a group, N is a near-field. □

Corollary 4[3] *Let R be a ring, and $E = \{e \in R | er = r \ \text{ for all } \ r \in R\}$. If E is nonempty and for each $r \in R, r \neq 0$, there is at least one element r' in R such that $rr' \in E$, then R is a division ring.*

References

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拟环为拟除环的几个条件

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摘　　要

我们定义了左、右不变拟子环和完全亚直既约拟环, 给出了拟环为拟除环的几个条件. 这些结果可以直接推广到结合环.

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Some Conditions for Near-ring to Be Near-field*

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All near-rings here are right near-rings. Let N be a near-ring. A subgroup M of $(N,+)$ is called a right(left) invariant subnear-ring of N if $MN \subseteq M$ $(NM \subseteq M)$. If M is both a left and a right invariant subnear-ring, then M is an invariant subnear-ring [4].

Proposition 1 *Let e be a distributive idempotent element of a zero-symmetric near-ring N. If Ne is a minimal left invariant subnear- ring of N, then eNe is a near-field.*

Proof If $n \in eNe$ and $n \neq 0$, then $n \in Ne$ and $Nn = Ne$. From $eNn = eNe$ we have $(eNe)n = eNe$. There is at least one element n' in eNe such that $n'n = e$. Thus eNe is a near-field. □

Proposition 2 *Let e be a distributive idempotent element of a zero-symmetric near-ring N. If N has no non-zero nilpotent left invariant subnear-rings and eNe is a near-field, then Ne is a minimal left invariant subnear-ring.*

Proof Let $A = Ne$. Assume L is a non-zero left invariant subnear- ring of N such that $L \subseteq A$. If $eL = \{0\}$, then $L^2 \subseteq AL = \{0\}$, a contradiction. Thus L has a non-zero element ene. There is $eme \in eNe$ such that $(eme)\cdot(ene) = e \in L$. Thus Ne is a minimal left invariant subnear-ring. □

Proposition 3 *If L is a minimal left invariant subnear-ring of a zero-symmetric near-ring N, then $L^2 = \{0\}$ or L has an idempotent element.*

Proof Suppose $L^2 \neq \{0\}$. Then L has an element n such that $Ln \neq \{0\}$ and $Ln = L$. Thus L has e such that $en = n$, so $e^2n = en$ and $(e^2 - e)n = 0$. Let $A = \{x \in L | xn = 0\}$. Then A is a left invariant subnear-ring of N and $A \neq L$. Hence $A = \{0\}, e^2 - e = 0$, i.e., L has an idempotent element e. □

Definition *Let N be a zero-symmetric near-ring. H is the intersection of all non-zero invariant subnear-rings of N. If $H \neq \{0\}$, N is called completely subdirect irreducible.*

Theorem 1 *Let e be a distributive idempotent element of a completely subdirect irreducible near-ring N and Ne be a minimal left invariant subnear-ring of N. If H has no non-zero nilpotent elements, then N is a near-field.*

Proof Let D be the left annihilator of H in N. If $eH = \{0\}$, then $e \in D$. Thus D is a

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non-zero invariant subnear-ring of N, so $H \subseteq D$ and $H^2 \subseteq DH = \{0\}$, a contradiction. Thus $eH \neq \{0\}$. If $He = \{0\}$, then $(eH)^2 = \{0\}$ and $eH = \{0\}$, a contradiction. Thus $He \neq \{0\}$, so $Ne = He \subseteq H$, and $e \in H$. Let $x \in N$ and $xe = 0$. Then $(eNx)^2 = \{0\}$ and $eNx = \{0\}$. Hence $ex = 0$. Similarly $ex = 0$ implies $xe = 0$. The left and the right annihilator of e in N are coincident, which we denote by K. If $K \neq \{0\}$, then $H \subseteq K$ and $eH \subseteq eK = \{0\}$, a contradiction. Thus $K = \{0\}$. Let $n \in N$. Then $(ne - n)e = 0 = e(en - n)$. Hence $ne - n = 0 = en - n$. By Proposition 1, $N = eNe$ is a near- field. □

Let R be an associative ring, H be the intersection of all non-zero ideals of R. If $H \neq \{0\}$, R is said to be subdirectly irreducible.

Corollary 1 *Let e be an idempotent element of a subdirectly irreducible ring R and Re be a minimal left ideal of R. If H has no non-zero nilpotent elements, then R is a division ring.*

Theorem 2 *Let N be a completely subdirect irreducible near- ring. If N is commutative and H has no non-zero nilpotent elements, then N is a near-field.*

Proof By assumption, H is a minimal left invariant subnear-ring of N. Since $H^2 \neq \{0\}$, H has an idempotent element e by Proposition 3. From $Ne \neq \{0\}$ and $Ne \subseteq H$ we have $Ne = H$. By Theorem 1, N is a field. □

Corollary 2 *Let R be a commutative subdirectly irreducible ring. If H has no non-zero nilpotent elements, then R is a field.*

Theorem 3 *Let L be the intersection of all non-zero left invariant subnear-rings of a zero-symmetric near-ring N. If L has a distributive idempotent element and has no non-zero nilpotent elements, then N is a near-field.*

Proof By assumption, L has a distributive idempotent element e and $L = Ne$ is a minimal left invariant subnear-ring of N. eNe is a near- field by Proposition 1. Let K be the left annihilator of e in N. If $K \neq \{0\}$, then $Le \subseteq Ke = \{0\}$, a contradiction. Thus $K = \{0\}$. Let $n \in N$. Then $(ne - n)e = 0, ne = n$, i.e., e is the right identity of N. If $x \in N$ and $ex = 0$, then $(xNe)^2 = \{0\}, xNe = \{0\}$, so $xe = 0$. Thus $x = 0$. Since $e(en - n) = 0$ for all $n \in N, en - n = 0$, i.e., e is the left identity of N. Thus $N = eNe$ is a near-field. □

Theorem 4 *Let L be the intersection of all non-zero left invariant subnear-rings of a zero-symmetric near-ring N. If L has a distributive idempotent element e such that the right annihilator of e in N is a left invariant subnear-ring of N, then N is a near-field.*

The proof is similar to that of Theorem 3, and is omitted.

Corollary 3[2] *Let L be the intersection of all non-zero left ideals of a ring R. If $L^2 \neq \{0\}$, then R is a division ring.*

Proof From $L^2 \neq \{0\}$, we get by Proposition 3 that L has an idempotent element e. By imitating the proof of Theorem 3, it can be deduced that the left annihilator K of e in R is $\{0\}$ and e is the right identity of R. Let $x \in R$ and $ex = 0$. Then $(xR)^2 = (xRe)^2 = \{0\}$.

If $xR \neq \{0\}$, then R has a non-zero nilpotent ideal I. That is $I^n = \{0\}$, where $n > 1$ is a fixed integer. Since $L \subseteq I$, $L^n = \{0\}$. This is a contradiction. Thus $xR = \{0\}$, so $xe = 0$ and $x \in K$. This proves the right annihilator of e in R is $\{0\}$ which is an ideal of R. By Theorem 4, R is a division ring. □

Theorem 5 *Let N be a zero-symmetric near-ring and $E = \{e \in N | en = n \text{ for all } n \in N\}$. If E is nonempty and for each $n \in N, n \neq 0$, and there is at least one element n' in N such that $nn' \in E$, then N is a near-field.*

Proof It is easily seen that $0 \notin E$ except $N = \{0\}$. If there are two different elements e, e' in E, there is at least one element n in N such that $(e - e')n \in E$, and $er = r, e'r = r$ for all $r \in N$. Hence $(e - e')r = 0$ for all $r \in N$, a contradiction. Thus $e = e'$. Let $n \in N$ and $n \neq 0$. There is at least one element m in N such that $nm = e \in E$. Hence $n(mn) = (nm)n = en = n$. There is at least one element k in N such that $(mn)k = e$. From $n(mn) = n$, we have $m[n(mn)] = (mn)(mn) = mn$. Multiplying the last equality from the right by k, we have $(mn - e)e = 0$. If $mn - e \neq 0$, there is at least one element h in N such that $(mn - e)h = e$. Multiplying $(mn - e)e = 0$ from the right by h, we have $e = 0$, a contradiction. Thus $mn = e$. Therefore $N \setminus \{0\}$ is a group, N is a near-field. □

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References

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