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Ornstein-Uhlenbeck 型马氏过程局部时与占位时的存在性

王永进
(南开大学数学系，天津 300071)

摘　要

本文考虑Ornstein-Uhlenbeck 型马氏过程的局部时，证明了在一定情形下局部时的存在性，并给出了不存在的反例，同时讨论了这类过程的占位时，指出了在某些限制性条件下，占位时密度的平方可积性

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The Existence of Local Times and Occupation Times for Ornstein- Uhlenbeck Type Markov Processes*

Wang Yongjin
(Dept. of Math., Nankai Univ., Tianjin)

Abstract The local times are considered for Ornstein-Uhlenbeck type Markov processes. In some cases the existence of local time for the process has been demonstrated. Meanwhile, occupation times have also been introduced and their existence for this class of processes will finally be revealed in some special situations

Key words $\alpha-$stable process, Ornstein-Uhlenbeck type Markov process, local time, occupationtime, potential density.

1. Introduction

Let X_t, $t \geq 0$ be an Ornstein-Uhlenbeck type Markov process with Lévy process A_t (abbr. O-U. M.P) (see Shiga [7]). That is

$$X_t = -\gamma \int_0^t X_s \, ds + A_t + x_0.$$

Where, A_t has Levy representation

$$Ee^{i\lambda A_t} = e^{-t\psi(\lambda)},$$

$$\psi(\lambda) = ia\lambda + 1/2b^2\lambda^2 - \int_{-\infty}^{\infty} (e^{i\lambda x} - 1 - \frac{i\lambda x}{1+x^2})\sigma(dx),$$

$(-\infty < a < \infty, b \geq 0)$ and Lévy measure $\sigma(\cdot)$ satisfies

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^2}\sigma(dx) < \infty.$$

As we know, if $b \neq 0$, process A_t is called having a Gaussian component. One class of typical processes having no Gaussian component are $\alpha-$stable process $(0 < \alpha < 2)$.

With these characters of the processes, we turn to their local times.

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The concept of local time was originally due to Lévy, he defined for Brownian motion the local time at x up to time t, $L(x,t)$, by

$$L(x,t) = \lim_{\epsilon \to 0} \frac{\text{meas}\{s : 0 \le s \le t, |X_s - x| < \epsilon\}}{2\epsilon}.$$

And later local time has become a standard concept defined by Blumenthal and Getoor [1]. In [3] Boylon demonstrated the existence and smoothes of local time for a large class of Markov processes, including some special processes, e.g., $\alpha-$stable process, $1 < \alpha < 2$. But for our present processes, what is the circumstance? The following result gives a partially positive answer.

2. The existence of local times

Theorem 1 *Assume that A_t either has a Gaussian component, or is an $\alpha-$stable process, $1 < \alpha < 2$. Then O-U. M.P with A_t admits a local time.*

The proof is not difficult, but it is necessary to quote two facts from [1] (or [2]).

The first one is a criterion for the existence of local time, and the second is a sufficient condition.

Lemma 1 *For a standard process X (defined in [1]), a necessary and sufficient condition that there exists a local time for X at x_0 is that x_0 is regular for $\{x_0\}$.*

For the proof of the theorem, the following lemma will offer us a method to prove that a single point is regular for its own point set.

Lemma 2 *If a standard process X, satisfying Hunt hypothese (F), has a $\beta-$potential density $u^\beta(x,y)$, for some $\beta > 0$, and assume that $u^\beta(x,x_0)$ is continuous at $x = x_0$, and is bounded for $x \in R$. Then x_0 is regular for $\{x_0\}$.*

Remark 1 *The processes approached in this paper, as concrete Markov processes, are standard processes (under B-G [1]'s meaning) satisfying Hunt hypothese (F).*

Now let us turn to the proof of Theorem 1.

Under the assumption of Theorem 1, by the calculations in [7], we know that there exists a measurable transition density function $p(t,x,y)$ for x, y. We proceed separately in two cases.

Case 1 A_t has a Gaussian component.

In this case, it is known that X_t has a jointly continuous transition density function $p(t,x,y)$ for x and y with some estimations (see [8]). Then

$$u^\beta(x,x_0) = \int_0^\infty e^{-\beta t} p(t,x,x_0)\, dt \le \int_0^\infty e^{-\beta t} 1/\sqrt{\pi b^2(1-e^{-2\gamma t})/\alpha}\, dt = c_1.$$

Where c_1 is some positive number. Plainly $u^\beta(x,x_0)$ is bounded on R, and is continuous in x by the dominated convergence theorem.

Case 2 A_t is an $\alpha-$stable process, $1 < \alpha < 2$.

Suppose that $p_\alpha(t,x,y)$ is transition density function.

$$u^\beta(x,x_0) \;=\; \int_0^\infty e^{-\beta t} p_\alpha(t,x,x_0)\, dt$$

$$
\begin{aligned}
&= \int_0^\infty e^{-\beta t} 1/(2\pi) \int_{-\infty}^\infty e^{-i\lambda x_0} \Psi_{x,t}(\lambda)\, d\lambda\, dt \\
&\quad (\text{where } \Psi_{x,t}(\lambda) = E_x e^{i\lambda X_t}) \\
&\le 1/\pi \int_0^\infty e^{-\beta t} \int_0^\infty e^{-\frac{c}{\alpha\gamma}(1-e^{-\alpha\gamma t})\lambda^\alpha}\, d\lambda\, dt \\
&\quad (\text{by the representation of } \Psi_{x,t}(\lambda) \text{ in [8], } c>0) \\
&\le 1/\pi \int_0^\infty d\lambda \int_0^1 e^{-\beta t} e^{-\frac{c}{\alpha\gamma}\beta_1\alpha\gamma t\lambda^\alpha}\, dt + 1/\pi \int_0^\infty d\lambda \int_1^\infty e^{-\beta t} e^{-\frac{c}{\alpha\gamma}(1-e^{-\alpha\gamma})\lambda^\alpha}\, dt \\
&\quad (\text{since there exists a } \beta_1 > 0 \text{ such that } 1-e^{-\alpha\gamma t} \ge \beta_1\alpha\gamma t, t \in (0,1)) \\
&= 1/\pi \int_0^\infty \frac{1}{\beta + c\beta_1\lambda^\alpha}\, d\lambda + \frac{e^{-\beta}}{\pi\beta} \int_0^\infty e^{-\frac{c}{\alpha\gamma}(1-e^{-\alpha\gamma})\lambda^\alpha}\, d\lambda \\
&= c_2(\beta,\alpha).
\end{aligned}
$$

Obviously, $c_2(\beta,\alpha)$ is finite, for $\beta > 0, 1 < \alpha < 2$. So for the same reason as in Case 1, we get to know that $u^\beta(x,x_0)$ satisfies the assumtion of Lemma 2.

Hence, either in Case 1 or in Case 2, by Lemma 1 and Lemma 2, there exists a local time for process X_t. This completes the proof of Theorem 1.

We have revealed the existence of local time for the process X under some conditions. Can any O-U. M.P admit a local time? The answer is negative, and the next proposition gives a counterexample.

Proposition 1 *Let A_t be such a Lévy process with triple parameter (a,b,σ) satisfying that*

$$\inf\{p>0, \int_{|x|\le 1} |x|^p \sigma(dx) < \infty\} < 1$$

and

$$\int_{-\infty}^\infty \frac{x}{1+x^2}\sigma(dx) \ne -a.$$

Then O-U.M.P with Lévy process A_t has no local time at $x_0 = 0$.

Remark 2 *A_t is usually regarded as starting at 0 and having almost all right continuous smaple paths with left limits.*

Proof By the decomposition of Lévy process, it follows that

$$A_t = -a't + J_t^+ - J_t^-.$$

Where,

$$
\begin{aligned}
a' &= a + \int_{-\infty}^\infty \frac{x}{1+x^2}\sigma(dx), \\
J_t^+ &= \sum_{s\le t} (A_s - A_{s-}) I_{(A_s - A_{s-} > 0)}, \\
J_t^- &= -\sum_{s\le t} (A_s - A_{s-}) I_{(A_s - A_{s-} < 0)}.
\end{aligned}
$$

By the assumption of Proposition 1, $a' \neq 0, x_0 = 0$.

$$X_t = -a'(1 - e^{-\gamma t})/\gamma + \int_0^t e^{-\gamma(t-s)} dJ_t^+ - \int_0^t e^{-\gamma(t-s)} dJ_t^- .$$

Since

$$\beta = \inf\{p > 0 : \int_{|x|<1} |x|^p \sigma(dx) < \infty\} < 1.$$

It implies that $\lim_{t\to 0} t^{-1} J_t^+ = 0$, and $\lim_{t\to 0} t^{-1} J_t^- = 0$, see [5].

So

$$\lim_{t\to 0} t^{-1} \int_0^t e^{-\gamma(t-s)} dJ_t^+ = 0, \quad \lim_{t\to 0} t^{-1} \int_0^t e^{-\gamma(t-s)} dJ_t^- = 0.$$

Then it follows that $|X_t| \geq |a'|t/2 \neq 0$, for sufficiently small t, a.s., since $a' \neq 0$. That is

$$\lim_{\delta \downarrow 0} P_0(X_t = 0, \quad \text{for some} \quad 0 < t < \delta) = 0.$$

It also means that 0 is not regular for $\{0\}$ and by Lemma 1, there exist no local time at $x_0 = 0$ for the process.

Remark 3 *$\alpha-$stable process $(0 < \alpha < 1)$ satisfies the condition of Proposition 1, and so there exists no local time at $x_0 = 0$, for the corresponding O-U.M.P.*

3. Occupation times for the processes

In this section, we introduce the occupation time to O-U.M.P. As in [4], the occupation measure of X up to time t is

$$\mu_t(\Gamma) = \lambda\{s \leq t : \ X(s) \in \Gamma\}$$

for Γ being a Borel set on R, λ being Lebesgue measure. With the regularity of process X, one has

$$\mu_t(\Gamma) = \int_0^t I_\Gamma(X_s)\, ds.$$

The main object of this section is to obtain some properties about occupation times for the processes. Here on, assume that A_t is an $\alpha-$stable process, $1 < \alpha \leq 2$.

Consider the occupation measure $\mu_t(\Gamma)$ of X, let

$$\mu^1(\Gamma) = \int_0^\infty e^{-t} \mu_t(\Gamma)\, dt.$$

Then

$$\mu^1(\Gamma) = \int_0^\infty e^{-t} I_\Gamma(X_t)\, dt.$$

First of all, consider the Fourier transform of measure μ^1, denoted by $\bar{\mu}^1$. We perform some calculations as follows

$$E_{x_0}|\bar{\mu}^1(\lambda)|^2 \;=\; E_{x_0}(\int_0^\infty e^{-s} e^{i\lambda X_s}\, ds)(\int_0^\infty e^{-t} e^{-i\lambda X_t}\, dt)$$

$$
\begin{aligned}
&= E_{x_0}\int_0^\infty\int_0^\infty e^{-(s+t)}e^{i\lambda(X_s-X_t)}\,dt\,ds\\
&= \int_0^\infty\int_0^\infty e^{-(s+t)}E_{X_s}[e^{-i\lambda X_t}E_{X_t}e^{i\lambda X_{s-t}}I_{(s\geq t)}e^{i\lambda X_s}E_{X_s}e^{-i\lambda X_{t-s}}I_{(t\geq s)}]\,ds\,dt\\
&\quad \text{(by Markov property)}\\
&\leq \int_0^\infty\int_0^\infty e^{-(s+t)}\exp\{-c\frac{1-e^{-\alpha\gamma t}}{\alpha\gamma}(1-e^{-\gamma(s-t)})^\alpha\lambda^\alpha\\
&\quad -\frac{c}{\alpha\gamma}(1-e^{-\alpha\gamma(s-t)})\lambda^\alpha\}I_{(s\geq t)}\,dt\,ds\\
&\quad +\int_0^\infty\int_0^\infty e^{-(s+t)}\exp\{-c\frac{1-e^{-\alpha\gamma s}}{\alpha\gamma}(1-e^{-\gamma(t-s)})^\alpha\lambda^\alpha\\
&\quad -\frac{c}{\alpha\gamma}(1-e^{-\alpha\gamma(t-s)})\lambda^\alpha\}I_{(t\geq s)}\,dt\,ds\\
&\leq \beta_1 e^{-\beta_2\lambda^\alpha}+\frac{\beta_2}{1+\beta_4\lambda^\alpha},
\end{aligned}
$$

where $\beta_j, j = 1,2,3,4$ are some positive numbers. Hence if $1 < \alpha \leq 2$, $\bar{u}^1(\lambda)$ is square integrable almost everywhere.

By the property of Fourier transform of measure, $\mu^1(x) = \frac{\mu^1(\Gamma)}{dx}$ exists, this implies that $\mu_t(\cdot)$ has an occupation density $\mu_t(x)$ a.s. Combine with Plancherel's identity $\| \mu \|_2^2 = (2\pi)^{-1}\| \bar{\mu} \|_2^2$.

Then $\mu_t(x,\omega)$ is square integrable with respect to measure $\lambda \times P$. (λ is Lebesgue measure on R, P is probability measure on Ω).

Thus, it has in fact deduced the following result.

Theorem 2 *If X_t is an O.U.M.P with $\alpha-$stable process, $1 < \alpha \leq 2$. Then X_t has a square integrable occupation time density.*

As we known, in some cases, occupation time density (if exists) is different from the local time, although sometimes they have the equivalent versions. The counterexample is referred to [4]. Now we take a fact from [4] which reveals their relations.

Theorem 3(Geman and Horowitz) *Assume that any point x in R is regular to its own single point set $\{x\}$. Then exists a version of local time $(L_t(x))$ such that $\alpha_t(x) = g(x)L_t(x)$ is an occupation time density, for a finite, positive and measurable function $g(x)$.*

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