$m(n+2t-1)+1$. Set $P_m = v_0, v_1, \cdots, v_{m-1}$. Then $H = K_{m(n+2t-1)+1} - V(P_m)$ is a complete graph with order $m[(n-1)+2t-1]+1$. By induction hypothesis, it follows that $K_{(n-1)+4t} - tK_3 \subseteq H \cap B_2$. So we obtain $K_{n+4t} - tK_3 \subseteq \langle V(K_{(n-1)+4t} - tK_3) \cup \{v_0\}\rangle \cap B_2$.

The proof of Theorem 2.1 is completed. □

References

[1] R.J. Gould and M.S. Jacobson, *On the Ramsey number of tree versus graphs with large clique number*, J. Graph Theory, **7**(1983), 71—78.

[2] V. Chvátal, *Tree-complete graph Ramsey numbers*, J. Graph Theory, **1**(1977), 93.

[3] G. Chartrand, R.J. Gould and A.D. Polimeni, *On the Ramsey number of forests versus nearly complete graph*, J. Graph Theory, **2**(1980), 233—239.

[4] M.S. Jacobson, *On Various Extensions of Ramsey Theory*, Doctorial Dissertation, Emory University, 1980.

[5] V. Chvátal and F. Harary, *Generalized Ramsey theory for graph, III, small offdiagonal numbers*, Pacific J. Math., **41**(1972), 235—245.

[6] J.A. Bondy and Y.R.S. Murty, *Graph Theory with Applications*, Macmillan Press, London, 1976.

若干广义 Ramsey 数

黄国泰

(海南师范学院数学系，海口 571158)

摘　要

本文讨论了关于树对完全图删去一些相交的三阶路的广义Ramsey 数 $R(T_m, K_n - tP_3)$ 和关于路对完全图删去一些不相交的三阶完全图的广义Ramsey 数 $R(P_m, K_n - tK_3)$, 获得如下结果:

1. 如果 $m \geq 3, n \geq 3$, 那么

$$R(T_m, K_n - tP_3) = (m-1)(n-t-1)+1,\quad 0 \leq t \leq [\frac{n}{3}].$$

2. 若 $m \geq 4, n, t \geq 1$, 则

$$R(P_m, K_n - tK_3) = (m-1)(n+2t-1)+1.$$

从而，这两个结果部分地回答了1983 年R.J. Gould 和M.S. Jacobson 在[1] 中提出的未解决问题.

Journal of Mathematical Research & Exposition
Vol.14, No.2, May 1994

Some Generalized Ramsey Numbers *

Huang Guotai
(Dept. of Math., Hainan Teachers College, Hainan Island, China)

1. Introduction

For simple graphs G_1 and G_2, the generalized Ramsey number associated with G_1 and G_2 is defined to be the smallest positive integer p such that if $K_p = B_1 \times B_2$ is an arbitrary factorization of K_p(i.e., B_1 and B_2 have orders p and $E(B_1) \cup E(B_2)$ partitions $E(K_p)$, then $G_1 \subseteq B_1$, or $G_2 \subseteq B_2$. A (G_1, G_2)-blocking pattern of K_p is a factorization $K_p = B_1 \times B_2$ such that $G_1 \nsubseteq B_1$ and $G_2 \nsubseteq B_2$. For a vertex v in K_p, define

$$\begin{aligned} N_i(v) &= \{u \in V(K_p)/vu \in E(B_i)\}, i = 1,2. \\ N_i^*(v) &= N_i(v) \bigcup \{v\}, i = 1,2. \\ |N_i(v)| &= d_i(v), i = 1,2. \end{aligned}$$

Let G be a subgraph of K_p, and $B_i(G)$ denote the set of vertices such that $v \in B_i(G)$, implies

$$N_i(v) \bigcap V(G) = \emptyset, \tag{1}$$

if $v \in V(G) - B_i(G)$, then

$$N_i(v) \bigcap V(G) \neq \emptyset. \tag{2}$$

If $S \subseteq V(G)$, the subgraph induced by S, which we denote by $\langle S \rangle$, is the subgraph with vertes set S and edge set consisting of those edges of G incident to two elements of S. We denote by $G_1 - G_2$ the graph obtained by deleting the edges of G_2 from the graph G_1.

We shall need the following results.

Theorem A[2] *Let T_m be a tree of order m, and n be a positive integer. Then*

$$R(T_m, K_n) = (m-1)(n-1) + 1.$$

*Received Apr. 28, 1992.

Theorem B[1] *Let $m \geq 3, n \geq 6$, and T_m be a tree of order m with $T_m \neq K_{1,m-1}$ for $m \geq 4$. Then*

$$R(T_m, K_n - tK_2) = (m-1)(n-t-1)+1, \text{ for } 0 \leq t \leq [\frac{n-2}{2}].$$

Theorem C[2] *Let T_m be a tree of order $m(\geq 3)$, and $n \geq 4$. Then*

$$R(T_m, K_n - K_2) = (m-1)(n-2)+1.$$

Furthermore, we have $R(T_m, G) = (m-1)(n-2)+1$, for each graph G of order n with clique number $n-1$.

Theorem D[2] *If G and H are simple graphs, then*

$$R(G, H) \geq (\chi(G)-1)(c(H)-1)+1,$$

where $\chi(G)$ and $c(H)$ denotes the chromatic number of G and the order of the largest component of H, respectively.

Theorem E[2] *If P_m is a graph of order $m(\geq 4)$ and G_n is a graph of order $n+2$ with clique number $n(\geq 3)$, then*

$$R(P_m, G_n) = (m-1)(n-1)+1.$$

The purpose of this paper is to investigate the generalized Ramsey number of tree versus complete graph minus multiple copies of path and complete graph of order 3. These results will give an partial answer to the problems raised by Gould and Jacobson in [1].

2. On the generalized Ramsey numbers $R(T_m, K_n - tP_3)$

Lemma 1.1 *If $m \geq 3$, and $0 \leq t \leq 1$, then*

$$\begin{aligned} R(T_m, K_3 - tP_3) &\leq (m-1)(3-t-1)+1, & (3)\\ R(T_m, K_4 - tP_3) &\leq (m-1)(4-t-1)+1, & (4)\\ R(T_m, K_5 - tP_3) &\leq (m-1)(5-t-1)+1. & (5) \end{aligned}$$

Proof If $t = 0$, the results follow directly from Theorem B. If $t = 1$, it follows from $K_4 - P_3 \subseteq K_4 - K_2$ and $K_5 - P_3 \subseteq K_5 - K_2$ that $R(T_m, K_4 - K_3) \leq R(T_m, K_4 - K_2), R(T_m, K_5 - P_3) \leq R(T_m, K_5 - K_2)$. This finishes the proof. □

Theorem 1.2 *Let $m \geq 3, n \geq 3, T_m$ be a tree of order m with $T_m \neq K_{1,m-1}$ for $m \geq 4$. Then*

$$R(T_m, K_n - tP_3) = (m-1)(n-t-1)+1, \text{ for } 0 \leq t \leq [\frac{n}{3}].$$

Proof By Theorem D, $R(T_m, K_n - tP_3) \geq (m-1)(n-t-1)+1$. It suffices to show that

$$R(T_m, K_n - tP_3) \leq (m-1)(n-t-1)+1. \tag{6}$$

This follows immediately from Lemma 1 for $n \leq 5$. Since $K_n - tP_3 \subseteq K_n - tK_2$, we have $R(T_m, K_n - tP_3) \leq R(T_m, K_n - tK_2)$. By Theorem B the inequality (6) holds for $n \geq 6$. The proof is complete. □

In order to give the generalized Ramsey number $R(K_{1,m}, K_n - tP_3)$, we need the following lemma.

Lemma 1.3 *If $n \geq 1, 0 \leq t \leq n$, then*

$$\begin{aligned} R(K_{1,3}, K_{3n} - tP_3) &\leq 3(3n-t-1)+1, & (7)\\ R(K_{1,3}, K_{3n+1} - tP_3) &\leq 3(3n-t)+1, & (8)\\ R(K_{1,3}, K_{3n+2} - tP_3) &\leq 3(3n-t+1)+1. & (9) \end{aligned}$$

Proof We proceed by induction on n. By Lemma 1, the results are obvious for $n=1$. To complete the argument, we need only to show that for each $t, 0 \leq t \leq n+1$, there hold

$$\begin{aligned} R(K_{1,3}, K_{3n+3} - tP_3) &\leq 3(3n-t+2)+1, & (10)\\ R(K_{1,3}, K_{3n+4} - tP_3) &\leq 3(3n-t+3)+1, & (11)\\ R(K_{1,3}, K_{3n+5} - tP_3) &\leq 3(3n-t+4)+1. & (12) \end{aligned}$$

Assume that there is a $(K_{1,3}K_{3n+3}-tP_3)$-blocking pattern of $K_{3(3n-t+2)+1}$. Let $K_{3(3n-t+2)+1}$ $= B_1 \times B_2$ such that $K_{1,3} \not\subseteq B_1$ and $K_{3n+3}-tP_3 \not\subseteq B_2$. For $0 \leq t \leq n$, since by the hypothesis $R(K_{1,3}K_{3n+2}-tP_3) \leq 3(3n-t+1)+1$, then $K_{3n+2}-tP_3 \subset B_2$. Set $G = K_{3n+2}-tP_3)$, and $H = K_{3(3n-t+3)+1} - V(G)$. Then H is a complete graph of order $6n-3t+5$. If there exists $v_0 \in V(H)$ such that $N_1(v_0) \cap V(G) = \emptyset$, then $K_{3n+3}-tP_3 \subseteq \langle V(G) \cup \{v_0\}\rangle \cap B_2$. This proves (10). If there exists $v_0 \in V(H)$ such that $N_1(v_0) \cap V(G) = \{v'\}$, and $v' \in B_1(G)$, then $K_{3n+2}-tP_3 \subseteq \langle V(G) \cup \{v_0\}\rangle \cap B_2$. This also finishes the proof of (10). Now suppose that there exists $v_0 \in V(H)$ such that $N_1(v_0) \cap V(G) = \{v'\}$, but $v' \notin B_1(G)$. Let v_1 be the central vertex of $P_3 (P_3 \subseteq B_1)$ and $v' \in V(P_3)$. Then $K_{3n+2} - tP_3 \subseteq \langle V(G) - \{v_1\}\rangle \cap B_2$. Hence, we may assume that for any $v \in V(H)$, there holds $|N_1(v) \cap V(G)| \geq 2$. It follows that $H \subseteq B_2$. Take $v_0 \in V(G)$, then $K_{3n+3} - tP_3 \subseteq \langle V(H) \cup \{v_0\}\rangle \cap B_2$, since $6n-3t+5 \geq 3n+2$. This contradicts the fact that $K_{3(3n-t+2)+1}$ is a $(K_{1,3}K_{3n+3}-tP_3)$-blocking pattern. For $t = n+1$, since $3(3n-t+2)+1 > 3(3n-t)+1$, $K_{3n+3}-tP_3 \subseteq B_2$. This contradiction gives the proof of the inequality (10).

Suppose $K_{3(3n-t+3)+1} = B_1 \times B_2$ is a $(K_{1,3}, K_{3n+4} - tP_3)$-blocking pattern. We take $v_0 \in V(K_{3(3n-t+3)+1})$, and set $H = K_{3(3n-t+3)+1} - N_1^*(v_0)$. Then H is a complete graph of order $3(3n-t+3)+1$. By inequality (9) we see that $K_{3n+3} - tP_3 \subseteq B_2 \cap H$, hence $K_{3n+4} - tP_3 \subseteq B_2 \cap (H + v_0)$. This contradiction gives the proof of (11).

By a similar argument, one can prove (12). □

Theorem 1.4 *If $m \geq 3, n \geq 3$, then $R(K_{1,m}, K_n - tP_3) = m(n-t-1)+1$, for $0 \leq t \leq [\frac{n}{3}]$.*

Proof From Theorem A and Theorem D, we need only to prove that for each $t, 1 \le t \le s$, there holds

$$R(K_{1,m}, K_{3s} - tP_3) \le m(3s - t - 1) + 1, \quad (13)$$
$$R(K_{1,m}, K_{3s+1} - tP_3) \le m(3s - t) + 1, \quad (14)$$
$$R(K_{1,m}, K_{3s+2} - tP_3) \le m(3s - t + 1) + 1. \quad (15)$$

We prove these inequalities by a double induction on m and s.

For $m = 3, s \ge 1$, and $m \ge 3, s = 1$, it is trivial from Lemma 1.1 and Lemma 1.3. Suppose the inequalities hold for $m \ge 3$ and $s \ge 1$. We prove the results for $m+1$ and $s+1$.

Let $K_{(m+1)(3s-t+2)+1} = B_1 \times B_2$, and $K_{1,m+1} \not\subseteq B_1$. By induction hypothesis, we have $K_{3s+2} - (t-1)P_3 \subseteq B_2$. Set $G = K_{3s+2} - (t-1)P_3$ and $H = K_{(m+1)(3s-t+2)} - V(G)$. Then H is a complete graph of order $m(3s - t + 2) - t + 1$. We proceed similarly to the proof of Lemma 1.2. Assume that there exists $v \in H$, such that $|N_1(v) \cap V(G)| \ge 2$. Since $s + 1 \ge t, m(3s - t + 2) - t + 1 \ge (m-1)(3s - t + 2) + 1$. By induction hypothesis again, we have $K_{3s+3} - tP_3 \subseteq B_2$. This gives the proof of the inequality (13).

Assume that $K_{(m+1)(3s-t+3)+1} = B_1 \times B_2$, and $K_{1,m+1} \not\subseteq B_1$. Choose a vertex v_0 from $K_{(m+1)(3s-t+2)+1}$. Then $K_{(m+1)(3s-t+2)+1} - N_1^*(v_0)$ is a complete graph of order at least $(m+1)(3s - t + 2) + 1$. It follows from (13) that $K_{3s+3} - tP_3 \subseteq B_2$, hence $K_{3s+4} - tP_3 \subseteq (K_{(m+1)(3s-t+3)+1} - N_1(v_0)) \cap B_2$, and the inequality (14) follows. Finally, the inequality (15) can be proved in an analoguous way. □

In summary, we have

Corollary 1.5 *If $m \ge 3, n \ge 3$, then $R(T_m, K_n - tP_3) = (m-1)(n - t - 1) + 1$, for each $t, 0 \le t \le [\frac{n}{3}]$.*

3. On the generalized Ramsey number $R(P_m, K_n - tK_3)$

Theorem 2.1 *If $m \ge 4$ and $n, t \ge 1$, then $R(P_m, K_{n+4t} - tK_3) = (m-1)(n + 2t - 1) + 1$.*

Proof It follows from Theorem D that $R(P_m, K_{n+4t} - tK_3) \ge (m-1)(n + 2t - 1) + 1$. By Theorem E, we see that for $m \ge 4$ and $n \ge 1$, there holds

$$R(P_m, K_{n+4} - K_3) \le (m-1)(n+1) + 1. \quad (16)$$

Now we show that

$$R(P_4, K_{1+4t}) - tK_3) \le 6t + 1, \text{ for } t \ge 1. \quad (17)$$

We prove the inequality (17) by induction on t.

The case of $t = 1$ is obvious from (16). Assume that for $t + 1 \ge 1$, the inequality (17) is true. Let $K_{6t+1} = B_1 \times B_2$. If $P_4 \not\subseteq B_1$, we are led to consider two cases:

Case 1. There exists $v_0 \in K_{6t+1}$ such that $d_1(v_0) \ge 4$. Set $N_1(v_0) = \{v_1, v_2, v_3, v_4, \cdots\}$, and $H = K_{6t+1} - \{v_0, v_1, \cdots, v_4\}$. Then H is a complete graph of order $6(t-1)+2$. By induction hypothesis we have $K_{1+4(t-1)} - (t-1)K_3 \subseteq B_2 \cap H$, and $N_1(v_i) \cap V(H) = \emptyset, i = 1, 2, 3, 4, v_iv_j \in E(B_2), 1 \le i \ne j \le 4$. Thus $K_{1+4t} - tK_3 \subseteq \langle V(K_1 + 4(t-1) - (t -$

$1)K_3) \cup \{v_1, v_2, v_3, v_4\}\rangle \cap B_2$.

Case 2. If for any $v \in K_{6t+1}$, there holds $d_1(v_0) \leq 3$, then we consider the following three subcases:

Subcase 1. There exists $v_0 \in K_{6t+1}$, such that $d_1(v_0) \leq 3$. Then we take $v \in K_{6t+1} - N_1^*(v_0)$ such that $d_1(v)$ is the largest, and set $N_1(v_0) = \{v_1, v_2, v_3\}, N_1(v) = \{v_1', \cdots\}$, and $H = K_{6t+1} - N_1^*(v_0) \cup \{v_1'\}$. Thus H is a complete graph of order $6(t-1)+1$. By induction hypothesis it follows that $K_{1+4(t-1)} - (t-1)K_3 \subseteq B_2$, so that

$$K_{1+4t} - tK_3 \subseteq \langle V(K_{1+4(t-1)} - (t-1)K_3) \bigcup \{v_1, v_2, v_3, v_1'\}\rangle \bigcap B_2.$$

Subcase 2. There exists $v_0 \in K_{6t+1}$, such that $d_1(v_0) = 2$. Take $v \in K_{6t+1} - N_1^*(v_0)$, and set $N_1(v_0) = \{v_1, v_2\}, N_1(v) = \{v_1', \cdots\}$, and $H = K_{6t+1} - N_1^*(v_0) \cup N_1^*(v)$. Then H is a complete graph of order at least $6(t-1)+1$. It follows from the induction hypothesis that $K_{1+4(t-1)} - (t-1)K_3 \subseteq B_2$. So we have

$$K_{1+4t} - tK_3 \subseteq \langle V(K_{1+4(t-1)} - (t-1)K_3) \bigcup N_1^*(v_0) \bigcup \{v_1'\}\rangle \bigcap B_2.$$

Subcase 3. For any $v \in K_{6t+1}$, there holds $d_1(v) \leq 1$. Then $K_{3t+1} \subseteq B_2$. Take $v_1, \cdots, v_t \in K_{6t+1} - V(K_{3t+1})$. It follows that

$$K_{1+4t} - tK_3 \subseteq \langle V(K_{3t+1}) \bigcup \{v_1, \cdots, v_t\}\rangle \bigcap B_2.$$

The proof of the inequality (17) is completed.

Now, we show that for $n \geq 1$ and $t \geq 1$,

$$R(P_4, K_{n+4t} - tK_3) \leq 3(n+2t-1)+1. \tag{18}$$

This is easy for $n = 1$ from the inequality (17). By the induction hypothesis, we assume that the inequality (18) is valid for $n \geq 1$ and fixed $t \geq 1$. We shall show that the inequality (18) is valid for $n+1$ and $t \geq 1$.

Let $K_{3(n+2t+1)} = B_1 \times B_2$. Assume that $P_4 \nsubseteq B_1$. If there exists $v_0 \in K_{3(n+2t+1)}$ such that $d_1(v_0) \geq 2$, set $H = K_{3(n+2t+1)} - N_1^*(v_0)$. Since H is of order at least $3[(n-1)+2t-1]+1$, it follows from the induction hypothesis that $K_{(n-1)+4t} - tK_3 \subseteq H \cap B_2$. So that $K_{n+4t} - tK_3 \subseteq \langle V(K_{(n-1)+4t} - tK_3) \cup \{v_0\}\rangle \cap B_2$. If there exists $v \in K_{3(n+t-1)+1}$ such that $d_1(v) \geq 3$, then take $v_0 \in K_{3(n+2t-1)+1}$, and set $N_1(v_0) = \{v_1, v_2, v_3, \cdots\}$. Thus $H = K_{3(n+2t-1)+1} - \{v_0, v_1, v_2\}$ is a complete graph of order $3[(n-1)+2t-1]+1$. By induction hypothesis, we see that $K_{(n-1)+4t} - tK_3 \subseteq H \cap B_2$, and $N_1(v_i) \cap V(H) = \emptyset$, for $i = 1, 2$. Therefore we have $K_{n+4t} - tK_3 \subseteq B_2$.

Finally we show that

$$R(P_m, K_{n+4t} - tK_3) \leq (m-1)(n+2t-1)+1, \text{ for } m \geq 4, n, t \geq 1. \tag{19}$$

By the inequality (18), (19) is trivial for $m = 4$. Now assume that (19) holds for $m \geq 4, n, t \geq 1$. Let $K_{m(n+2t-1)+1} = B_1 \times B_2$, then $P_m \neq B_1$, since $R(P_m, K_{n+4t} - tK_3) \leq$

$m(n+2t-1)+1$. Set $P_m = v_0, v_1, \cdots, v_{m-1}$. Then $H = K_{m(n+2t-1)+1} - V(P_m)$ is a complete graph with order $m[(n-1)+2t-1]+1$. By induction hypothesis, it follows that $K_{(n-1)+4t} - tK_3 \subseteq H \cap B_2$. So we obtain $K_{n+4t} - tK_3 \subseteq \langle V(K_{(n-1)+4t} - tK_3) \cup \{v_0\}\rangle \cap B_2$.

The proof of Theorem 2.1 is completed. □

References

[1] R.J. Gould and M.S. Jacobson, *On the Ramsey number of tree versus graphs with large clique number*, J. Graph Theory, **7**(1983), 71—78.

[2] V. Chvátal, *Tree-complete graph Ramsey numbers*, J. Graph Theory, **1**(1977), 93.

[3] G. Chartrand, R.J. Gould and A.D. Polimeni, *On the Ramsey number of forests versus nearly complete graph*, J. Graph Theory, **2**(1980), 233—239.

[4] M.S. Jacobson, *On Various Extensions of Ramsey Theory*, Doctorial Dissertation, Emory University, 1980.

[5] V. Chvátal and F. Harary, *Generalized Ramsey theory for graph, III, small offdiagonal numbers*, Pacific J. Math., **41**(1972), 235—245.

[6] J.A. Bondy and Y.R.S. Murty, *Graph Theory with Applications*, Macmillan Press, London, 1976.

若干广义Ramsey数

黄国泰

(海南师范学院数学系，海口571158)

摘　要

本文讨论了关于树对完全图删去一些相交的三阶路的广义Ramsey数$R(T_m, K_n - tP_3)$和关于路对完全图删去一些不相交的三阶完全图的广义Ramsey数$R(P_m, K_n - tK_3)$，获得如下结果:

1. 如果$m \geq 3, n \geq 3$，那么

$$R(T_m, K_n - tP_3) = (m-1)(n-t-1)+1, \quad 0 \leq t \leq [\frac{n}{3}].$$

2. 若$m \geq 4, n, t \geq 1$，则

$$R(P_m, K_n - tK_3) = (m-1)(n+2t-1)+1.$$

从而，这两个结果部分地回答了1983年R.J. Gould和M.S. Jacobson在[1]中提出的未解决问题.