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一类强藕合抛物型方程组整体解的存在性及渐近性态

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摘　要

本文把半群理论与能量方法结合起来，证明了一类强藕合非线性抛物型方程组的初边值问题解的整体存在性，并给出了解的$W^{2,p}$全局估计.

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Global Existence and Asymptotic Behaviour of Solution for a Strongly Coupled Parabolic System *

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1. Introduction

In this paper, we are interested in global existence and asymptotics for nonnegative solution of the following problem:

$$
\begin{array}{ll}
u_t = a \triangle u + b \triangle v + f(u,v) & \text{in } \Omega \times (0,+\infty) \\
v_t = c \triangle u + d \triangle v + g(u,v) & \\
u_t(x,t) = v(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty) \\
u(x,0) = u_0(x), v(x,0) = v_0(x) & \text{in } \Omega.
\end{array}
\tag{1.1}
$$

Here Ω is a bounded domain in R^N with the smooth boundary $\partial\Omega, u_0$ and v_0 are the given nonnegative functions.

Problem (1.1) arises in ecology as a model of two competing species with cross-diffusion effects (see [1,2]). In [2], M. Kirane considered the global bounds and asymptotics for problem (1.1) for $b = 0, a > d > 0$ and $f = -g$. In [3], we proved that there was no global solution for (1.1) if $f(u,v) + g(u,v) \geq k_0(|u|^p + |v|^p), (k_0 > 0, p > 1)$, and $\|u_0\|_{c(\bar\Omega)} \geq \delta_1, \|v_0\|_{c(\bar\Omega)} \geq \delta_1$, for some $\delta_1 > 0$.

The object of this paper is to prove the global existence and asymptotics if u_0 and v_0 are small in the L^p norm. The proof is given by combining the energy method and the linear semigroup theory.

The following assumption will be made throughout:

$$
(H)\left\{
\begin{array}{l}
(1).\ \text{the constants } a,b,c,d \text{ are such that } a,d > \dfrac{|b|+|c|}{2} \text{ and } bc \geq 0. \\
(2).\ f,g \in C^1 \text{ and there exist some positive constants } k_1,k_2,\alpha_i (i=1,2,3,4) \\
\qquad \text{such that} \\
\qquad\qquad |f(u,v)| \leq k_1(|u|^{1+\alpha_1} + |v|^{1+\alpha_2}) \quad \text{for } (u,v) \in R^2, \\
\qquad\qquad |g(u,v)| \leq k_2(|u|^{1+\alpha_3} + |v|^{1+\alpha_4}).
\end{array}
\right.
$$

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Now let us state the main results. Define $\alpha_0 = \max\limits_{1\le i\le 4}\{\alpha_i\}$.

Theorem 1. *Assume (H) and* $0 < \alpha_0 < \frac{4}{N}, \partial\Omega \in C^m$ *and* $\frac{N}{4m} < 1$. *Then there exist a constant* $d_0 > 0$, *such that if nonnegative* $u_0, v_0 \in L^2(\Omega)$ *and* $\|u_0\|_2^2 + \|v_0\|_2^2 < d_0$, *the problem (1.1) admits a unique nonnegative solution* (u,v) *which satisfies* $u,v \in C_{((0,+\infty),W_0^{1,2}\cap W^{2,2})} \cap C^1_{((0,+\infty),L^2)}$ *and for any* $T > 0$,

$$\|u(t)\|_{W^{2,2}} + \|v(t)\|_{W^{2,2}} \le C_1(T)(\|u_0\|_2 + \|v_0\|_2)e^{-b_1 t} \quad t \ge T$$

for some $b_1 > 0$.

If $b = c = 0$, the restriction on α_i in Theorem 1 can be removed.

Theorem 2. *Suppose (H) and* $b = c = 0$. $\partial\Omega \in C^{2,\alpha}, (0 < \alpha < 1)$. *Let* $p > \max\{\frac{N}{4}, \frac{\alpha_0 N}{2} - 2\}$. *Then there is* $d_1 > 0$, *such that if nonnegative* $u_0, v_0 \in L^{p+2}$ *and* $\|u_0\|_{p+2}^{p+2} + \|v_0\|_{p+2}^{p+2} < d_1$, *there exists a unique solution* (u,v) *of (1.1) which satisfies*

(1) $u, v \in C_{((0,+\infty),W_0^{1,p+2}\cap W^{2,2+p})} \cap C^1_{((0,+\infty),L^{p+2})}$.

(2) $\|u(t)\|_{W^{2,p+2}} + \|v(t)\|_{W^{2,p+2}} \le C_2(T)(\|u_0\|_{p+2} + \|v_0\|_{p+2})e^{-b_2 t}$ $\;t \ge T > 0$ *with* $b_2 > 0$.

Remark In Theorem 1, we need that the boundary $\partial\Omega$ is enough smooth to let imbedding $D(A^\beta) \subset C^r(\bar\Omega)$ be held. $(0 < r < 2m\beta - \frac{N}{2})$.

2. Auxiliar results

For $1 \le p < +\infty, \|\cdot\|_p$ will denote $L^p(\Omega)$ norm and

$$\begin{aligned} &D(A) = W_0^{1,p}(\Omega) \bigcap W^{2,p}(\Omega), \\ &Au = -\triangle u \qquad \forall u \in D(A). \end{aligned} \tag{2.1}$$

It is well known (see [4]) that A is a sectorial operator in L^p and e^{-tA} is the semigroup in L^p generated by A. Now, let us recall some results ([4]).

Lemma 1. *Let A be as above.*

(1) There exists $\delta > 0$, *such that for* $0 \le \beta \le 1$

$$\|A^\beta e^{-tA}\|_p \le M_\beta t^{-\beta} e^{-\delta t} \;\; (t > 0). \tag{2.2}$$

(2) If $0 \le \beta \le \alpha \le 1$, *then* $D(A^\alpha) \subset D(A^\beta)$.

(3) Let $\Omega \subset R^N$ *be a bounded domain with boundary* $\partial\Omega \in C^m$. *If* $0 \le \beta \le 1$, *then*

$$\begin{aligned} &D(A^\beta) \subset W^{k,q}(\Omega) \quad \text{for } k - \tfrac{N}{q} < 2m\beta - \tfrac{N}{p} \quad q \ge p, \\ &D(A^\beta) \subset C^r(\bar\Omega) \qquad \text{for } 0 \le r < 2m\beta - \tfrac{N}{p}. \end{aligned} \tag{2.3}$$

Define the operator $B : D(A) \times D(A) \to L^p \times L^p$ as folows:

$$BU(T) = (-aAu(t) - bAv(t), -cAu(t) - dAv(t)), \tag{2.4}$$

where $U(t)=(u(t),v(t)), t>0$. If we denote $F(U)=(f(u,v),g(u,v))$, we can convert problem (1.1) to an abstract Cauchy problem in Banach space $L^p\times L^p$:

$$\begin{aligned}&\frac{dU}{dt}=BU(t)+F(U(t)) \qquad t>0,\\ &U(0)=(u_0,v_0).\end{aligned} \tag{2.5}$$

Moreover, we define a family of operators $S(t): L^p\times L^p\to L^p\times L^p$

$$\begin{aligned}S(t)(u,v) \;=\; &(e^{-a_0tA}(b_0u+\frac{b}{a_0-a_1}v)+e^{-a_1tA}((1-b_0)u-\frac{b}{a_0-a_1}v),\\ &e^{-a_0tA}(\frac{c}{a_0-a_1}u+(1-b_0)v)+e^{-a_1tA}(\frac{-c}{a_0-a_1}u+b_0v)).\end{aligned} \tag{2.6}$$

Here

$$\begin{aligned}a_0 &= (a+d-\sqrt{(a-d)^2+4bc}\,)/2,\\ a_1 &= (a+d+\sqrt{(a-d)^2+4bc}\,)/2,\\ b_0 &= (d-a_0)/(a_1-a_0).\end{aligned} \tag{2.7}$$

Since $ad>(|b|+|c|)^2/4\ge|bc|\ge bc\ge0$, hence $a_0>0, a_1>0, 0\le b_0<1$. When $bc=0$ and $a=d$, we let $b_0=0$. It is obvious that

$$\begin{aligned}&b_0^2+\frac{bc}{(a_0-a_1)^2}=b_0, \quad (1-b_0)a_0+b_0a_1=d\\ &b_0a_0+(1-b_0)a_1=a.\end{aligned} \tag{2.8}$$

If $b=c=0$, we define

$$S(t)(u,v)=(e^{-atA}u,e^{-dtA}v). \tag{2.9}$$

For $S(t)$ $(t>0)$, we have

Lemma 2 *$S(t)$ $(t>0)$ is an analytic semigroup in $L^p\times L^p$ with the infinitesimal generator B.*

Proof It should be noted that the operators e^{-a_0tA} and e^{-a_1tA} are the linear semigroups in L^p generated by a_0A and a_1A respectively. It suffices to prove that for any $U=(u,v)\in D(A)\times D(A)$

$$\lim_{t\downarrow0}\frac{S(t)U-U}{t}=BU \qquad \text{in } L^p \tag{2.10}$$

and for any $U=(u,v)\in L^p\times L^p$

$$S(t+s)U=S(t)S(s)U, \qquad t,s\ge0. \tag{2.11}$$

In fact, we have

$$\lim_{t\downarrow0}t^{-1}(b_0e^{-a_0tA}u+(1-b_0)e^{-a_1tA}u-u)$$

$$\lim_{t\downarrow 0}(b_0\frac{e^{-a_0tA}u-u}{t})+(1-b_0)\lim_{t\downarrow 0}\frac{e^{-a_1tA}u-u}{t}$$
$$=-b_0a_0Au-(1-b_0)a_1Au=-aAu \quad \text{in } L^p,$$

and

$$\lim_{t\downarrow 0}\frac{b}{a_0-a_1}t^{-1}(e^{-a_0tA}v-e^{-a_1tA}v)$$
$$=\frac{b}{a_0-a_1}\lim_{t\downarrow 0}(\frac{e^{-a_0tA}v-v}{t}-\frac{e^{-a_1tA}v-v}{t})$$
$$=\frac{b}{a_1-a_0}(a_0Av-a_1Av)=-bAv \quad \text{in } L^p.$$

Similarly, we obtain

$$\lim_{t\downarrow 0}t^{-1}(e^{-a_0tA}(\frac{c}{a_0-a_1}u+(1-b_0)v)$$
$$+e^{-a_1tA}(\frac{-c}{a_0-a_1}u+b_0v)-v)=-(cAu+dAv) \quad \text{in } L^p.$$

Therefore (2.10) is true. Also, by direct computation, we see that (2.11) holds.

Lemma 3 *Let (H) be satisfied. Then, for any nonnegative $U(0)=(u_0,v_0)\in C_0^3(\Omega)\times C_0^3(\Omega)$, the Cauchy problem (2.5) has a unique solution $U(t)=(u(t),v(t))$, such that $u(t),v(t)\in C_{([0,T_{\max}),C^2(\Omega))}\cap C^1_{((0,T_{\max}),C(\Omega))}$. Moreover, if $T_{\max}<+\infty$, then*

$$\lim_{t\uparrow T_{\max}}(\|u(t)\|_0+\|v(t)\|_0+\|Au(t)\|_0+\|Av(t)\|_0)=+\infty. \tag{2.12}$$

Here $\|\cdot\|_0$ denotes $C(\bar\Omega)$ norm.

Proof From the assumption (H), $F(U)$ is continuously differentiable in $C(\bar\Omega)\times C(\bar\Omega)$. By virtue of Theorem 6.1.5 in [4], the initial value problem (2.5) admits a unique classical solution

$$U(t)=S(t)U(0)+\int_0^t S(t-s)F(U(s))ds \quad 0\le t<T_{\max} \tag{2.13}$$

and if $T_{\max}<+\infty$, then

$$\lim_{t\uparrow T_{\max}}(\|U(t)\|_0+\|BU(t)\|_0)=+\infty, \tag{2.14}$$

so (2.12) is true, where $U(t)=(u(t),v(t))$. By a standard argument (see [3]), we see that $(u(t),v(t))$ is nonnegative and $u(t),v(t)\in C^2(\Omega) \quad (t>0)$.

Also we need an inequality which can be regarded as the generalized Gagliardo- Nirenberg inequality (see Lemma 3. [5]).

Lemma 4 *For all u with $|u|^k u\in W^{1,p}(\Omega), k\ge 0, p>1$, we have*

$$\|u\|_q\le C\|u\|_r^{1-\theta}\|\nabla(|u|^k u)\|_p^{\theta/(1+k)} \tag{2.15}$$

with a constant C independent of Ω and with $\theta = (1+k)(N^{-1}-p^{-1}+(1+k)/r)^{-1}(r^{-1}-q^{-1})$ provided that $q \geq 1+k$ and

(i) $1 \leq r \leq q \leq (1+k)Np/(N-p)$ *if* $N > p$;

(ii) $1 \leq r \leq q \leq +\infty$ *if* $N = p > 1$;

(iii) $1 \leq r \leq q \leq +\infty$ *if* $1 \leq N < p$.

3. Proof of Theorem 1

In this section, we shall assume that all conditions in Theorem 1 are satisfied. Our starting point is the following estimates.

Lemma 5 *Let $(u(t), v(t))$ be a smooth solution of (1.1) with nonnegative $u_0, v_0 \in C_0^3(\Omega)$. Then there exists $d_0 > 0$, such that if $\|u(t)\|_2^2 + \|v(t)\|_2^2 < d_0$, we have for $0 \leq t < T_{\max}$*

$$\|u_0\|_2^2 + \|v_0\|_2^2 < d_0 \tag{3.1}$$

and

$$\|u(t)\|_2^2 + \|v(t)\|_2^2 \leq C(\|u_0\|_2^2 + \|v_0\|_2^2)e^{-r_1 t} \tag{3.2}$$

with $C, r_1 > 0$.

Proof Multiplying the first equation in (1.1) by $u(t)$, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx = -a\int_\Omega |\nabla u|^2 dx - b\int_\Omega \nabla u \nabla v dx + \int_\Omega f u dx. \tag{3.3}$$

Similarly,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 dx = -d\int_\Omega |\nabla v|^2 dx - c\int_\Omega \nabla u \nabla v dx + \int_\Omega g v dx. \tag{3.4}$$

Adding these identities and using Cauchy-Schwartz's inequality, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega (u^2+v^2)dx + (a - \frac{|b|+|c|}{2})\int_\Omega |\nabla u|^2 dx \\ &+(d - \frac{|b|+|c|}{2})\int_\Omega |\nabla v|^2 dx \leq \int_\Omega |fu + gv| dx.\end{aligned} \tag{3.5}$$

By virtue of Young's inequality, we have

$$|f\cdot u| \leq k_1(|u|^{1+\alpha_1} + |v|^{1+\alpha_2})|u| \leq k_1(|u|^{2+\alpha_1} + |v|^{2+\alpha_2} + |u|^{2+\alpha_2}) \tag{3.6}$$

and

$$|g\cdot v| \leq k_2(|u|^{2+\alpha_3} + |v|^{2+\alpha_3} + |v|^{2+\alpha_4}). \tag{3.7}$$

Because of $0 < \alpha_0 < \dfrac{4}{N}$, we can choose $r \in [1,2]$, such that $r > \dfrac{\alpha_0 N}{2}$. From Lemma 4, we have

$$\int_\Omega |u|^{2+\alpha_i} dx \leq C\|u\|_r^{(1-\theta_i)(2+\alpha_i)}\|\nabla u\|_2^{\theta_i(2+\alpha_i)} \tag{3.8}$$

with $\theta_i = (\frac{1}{r} - \frac{1}{2+\alpha_i})(\frac{1}{N} - \frac{1}{2} + \frac{1}{r})^{-1}$. It follows from the assumption on r that $\theta_i(2+\alpha_i) < 2$.

Let $\beta_i = 2 - \theta_i(2+\alpha_i)$. By Sobolev's inequality, we obtain

$$\|u\|_r^{(1-\theta_i)(2+\alpha_i)} = \|u\|_r^{\alpha_i}\|u\|_r^{\beta_i} \leq C\|u\|_2^{\alpha_i}\|u\|_2^{\beta_i} \leq C\|u\|_2^{\alpha_i}\| \nabla u\|_2^{\beta_i}. \tag{3.9}$$

Here and in the following we write C or C_i for various positive constants which are independent of u and v.

Combining (3.8) with (3.9) yields

$$\int_\Omega |u|^{2+\alpha_i} dx \leq C\|u\|_2^{\alpha_i}\| \nabla u\|_2^2. \tag{3.10}$$

Similarly,

$$\int_\Omega |v|^{2+\alpha_i} dx \leq C\|v\|_2^{\alpha_i}\| \nabla v\|_2^2. \tag{3.11}$$

If we denote $\lambda = 2\min\{a - \frac{|b|+|c|}{2}, d - \frac{|b|+|c|}{2}\}$, (3.5) can be written as follows

$$\begin{aligned}&\frac{d}{dt}(\|u\|_2^2 + \|v\|_2^2) + \lambda(\| \nabla u\|_2^2 + \| \nabla v\|_2^2)\\ &\leq C(\sum_{i=1}^{4}(\|u\|_2^{\alpha_i} + \|v\|_2^{\alpha_i}))(\| \nabla u\|_2^2 + \| \nabla v\|_2^2).\end{aligned} \tag{3.12}$$

Using ε-Young's inequality and the fact that $\alpha_i \leq \alpha_0 (i = 1, 2, 3, 4)$, we see that

$$C(\|u\|_2^{\alpha_i} + \|v\|_2^{\alpha_i}) \leq C_1(\|u\|_2^2 + \|v\|_2^2)^{\alpha_0/2} + \frac{\lambda}{8}.$$

Thus, we drive

$$\frac{d}{dt}\varphi(t) \leq (C_1\varphi^{\frac{\alpha_0}{2}}(t) - \frac{\lambda}{2})(\| \nabla u(t)\|_2^2 + \| \nabla v(t)\|_2^2), \tag{3.13}$$

where $g(t) = \|u(t)\|_2^2 + \|v(t)\|_2^2$, constant $t \geq 0, C_1 > 0$.

Now we make the assumption that

$$\varphi(0) = \|u_0\|_2^2 + \|v_0\|_2^2 < (\frac{\lambda}{2C_1})^{2/\alpha_0} = d_0 > 0. \tag{3.14}$$

Then, (3.14) and (3.13) implies that $\varphi(t)$ is a nonincreasing function and

$$\varphi(t) \leq \varphi(0), \quad 0 \leq t < T_{\max}.$$

Also, we find that

$$\frac{d\varphi}{dt} + \varepsilon_1(\| \nabla u(t)\|_2^2 + \| \nabla v(t)\|_2^2) \leq 0 \tag{3.15}$$

with $\varepsilon_1 = \frac{\lambda}{2} - C_1\varphi^{\frac{\alpha_0}{2}}(0) > 0$.

Since $u \in W_0^{1,2}(\Omega)$, we have (see [6])

$$\|u(t)\|_2^2 \le \frac{1}{\lambda_1}\| \nabla u(t)\|_2^2, \quad \|v(t)\|_2^2 \le \frac{1}{\lambda_1}\| \nabla v(t)\|_2^2, \tag{3.16}$$

where $\lambda_1 > 0$ is the first (smallest) eigenvalue of the problem

$$\begin{aligned} \Delta w + \lambda w = 0 &\quad \text{in } \Omega, \\ w = 0 &\quad \text{on } \partial\Omega. \end{aligned}$$

Applying (3.16) to the left side of (3.15), we get

$$\frac{d\varphi}{dt} + \varepsilon_1\lambda_1\varphi(t) \le 0. \tag{3.17}$$

It implies that

$$\varphi(t) \le \varphi(0)e^{-\varepsilon_1\lambda_1 t} \quad 0 \le t < T_{\max}.$$

That is

$$\|u(t)\|_2^2 + \|v(t)\|_2^2 \le (\|u_0\|_2^2 + \|v_0\|_2^2)e^{-r_1 t} \quad 0 \le t < T_{\max} \tag{3.2}$$

with $r_1 = \varepsilon_1\lambda_1 > 0$.

Next, using the estimate (3.2), we have

Lemma 6 *Under the assumption of Lemma 5 and* $\max\{\frac{1}{2}, \frac{N}{4m}, \frac{\alpha_0 N}{4}\} < \beta < 1$, *we get for any* $0 < T < T_{\max}$

$$\|A^\beta u(t)\|_2 + \|A^\beta v(t)\|_2 \le C_1(T)(\|u_0\|_2 + \|v_0\|_2)e^{-b_1 t} \quad 0 < T \le t < T_{\max}, \tag{3.18}$$

where m is a constant in Theorem 1 and $b_1 > 0$.

Proof At first, we note that (2.13) can be written as follows

$$\begin{aligned} u(t) = {} & e^{-a_0 tA}(b_0 u_0 + \frac{b}{a_0 - a_1}v_0) + e^{-a_1 tA}((1 - b_0)u_0 - \frac{b}{a_0 - a_1}v_0) \\ & + \int_0^t e^{-a_0(t-s)A}(b_0 f(u,v) + \frac{b}{a_0 - a_1}g(u,v))ds \\ & + \int_0^t e^{-a_1(t-s)A}((1 - b_0)f(u,v) - \frac{b}{a_0 - a_1}g(u,v))ds \quad t > 0, \end{aligned} \tag{3.19}$$

$$\begin{aligned} v(t) = {} & e^{-a_0 tA}(\frac{c}{a_0 - a_1}u_0 + (1 - b_0)v_0) + e^{-a_1 tA}(\frac{-c}{a_0 - a_1}u_0 + b_0 v_0) \\ & + \int_0^t e^{-a_0(t-s)A}(\frac{c}{a_0 - a_1}f(u,v) + (1 - b_0)g(u,v))ds \\ & + \int_0^t e^{-a_1(t-s)A}(\frac{c}{a_0 - a_1}f(u,v) + b_0 g(u,v))ds \quad t > 0. \end{aligned} \tag{3.20}$$

By (2.2), there exist $C > 0$ and $\delta > 0$ such that

$$\|A^\beta e^{-a_0 tA}(b_0 u_0 + \frac{b}{a_0 - a_1} v_0)\|_2 \leq Ce^{-a_0 \delta t}(\|u_0\|_2 + \|v_0\|_2)t^{-\beta}$$

and

$$\|A^\beta e^{-a_0(t-s)A}(b_0 f + \frac{b}{a_0 - a_1} g)\|_2 \leq C(t-s)^{-\beta} e^{-a_0\delta(t-s)}(\|f\|_2 + \|g\|_2).$$

Similarly, we have

$$\|A^\beta e^{-a_1 tA}((1-b_0)u_0 + \frac{b}{a_0 - a_1} v_0)\|_2 \leq Ce^{-a_1 \delta t}(\|u_0\|_2 + \|v_0\|_2)t^{-\beta}$$

and

$$\|A^\beta e^{-a_1(t-s)A}((1-b_0)f + \frac{b}{a_0 - a_1} g)\|_2 \leq C(t-s)^{-\beta} e^{-a_1\delta(t-s)}(\|f\|_2 + \|g\|_2).$$

Using these inequalities in (3.19), we obtain

$$\begin{aligned} \|A^\beta u(t)\|_2 \leq\; & C(\|u_0\|_2 + \|v_0\|_2)t^{-\beta} e^{-a_0\delta t} \\ & + C\int_0^t (t-s)^{-\beta} e^{-a_0\delta(t-s)}(\|f\|_2 + \|g\|_2)ds. \end{aligned} \tag{3.21}$$

The assumption (H) gives

$$\begin{aligned} \|f\|_2 &\leq 2k_1(\|u\|_{2(1+\alpha_1)}^{1+\alpha_1} + \|v\|_{2(1+\alpha_2)}^{1+\alpha_2}), \\ \|g\|_2 &\leq 2k_2(\|u\|_{2(1+\alpha_3)}^{1+\alpha_3} + \|v\|_{2(1+\alpha_4)}^{1+\alpha_4}). \end{aligned} \tag{3.22}$$

Now, we appeal to the following inequality (see [7] or [8])

$$\|u\|_{2(1+\alpha_i)} \leq C\|A^\beta u\|_2^{\theta_i}\|u\|_2^{1-\theta_i} \tag{3.23}$$

with $\theta_i = \dfrac{\alpha_i N}{4(1+\alpha_i)\beta}$. Since $\beta > \dfrac{\alpha_0 N}{4}$, we have $(1+\alpha_i)\theta_i = \dfrac{\alpha_i N}{4\beta} < 1$. By the imbedding $D(A^\beta) \subset W^{1,2}(\Omega), (\beta > \dfrac{1}{2})$, we find that

$$\begin{aligned} \|u\|_{2(1+\alpha_i)}^{1+\alpha_i} &\leq C\|u\|_2^{\alpha_i}\|u\|_2^{1-\frac{\alpha_i N}{4\beta}}\|A^\beta u\|_2^{\frac{\alpha_i N}{4\beta}} \\ &\leq C\|u\|_2^{\alpha_i}\|A^\beta u\|_2^{1-\frac{\alpha_i N}{4\beta}}\|A^\beta u\|_2^{\frac{\alpha_i N}{4\beta}} = C\|u\|_2^{\alpha_i}\|A^\beta u\|_2. \end{aligned} \tag{3.24}$$

There is a similar estimate for v. Thus from (3.2) and above inequalities, we can drive

$$\begin{aligned} \|A^\beta u(t)\|_2 \leq\; & C(\|u_0\|_2 + \|v_0\|_2)t^{-\beta} e^{-a_0\delta t} \\ & + C\int_0^t (t-s)^{-\beta} e^{-a_0\delta(t-s)} e^{-\delta_1 s}(\|A^\beta u(s)\|_2 + \|A^\beta v(s)\|_2)ds \end{aligned} \tag{3.25}$$

and

$$\|A^\beta v(t)\|_2 \le C(\|u_0\|_2+\|v_0\|_2)t^{-\beta}e^{-a_0\delta t} + C\int_0^t (t-s)^{-\beta}e^{-a_0\delta(t-s)}e^{-\delta_1 s}(\|A^\beta u(s)\|_2+\|A^\beta v(s)\|_2)ds \tag{3.26}$$

for some $\delta_1 > 0$.

If we let $\varphi_1(t) = \|A^\beta u(t)\|_2 + \|A^\beta v(t)\|_2$, then (3.25) and (3.26) give

$$\varphi_1(t) \le C(\|u_0\|_2+\|v_0\|)t^{-\beta}e^{-a_0\delta t} + C\int_0^t (t-s)^{-\beta}e^{-a_0\delta(t-s)}e^{-\delta_1 s}\varphi_1(s)ds. \tag{3.27}$$

This inequality implies, for any $0 < T < T_{\max}$ (see [7]),

$$\varphi_1(t) \le C_1(T)(\|u_0\|_2+\|v_0\|_2)e^{-b_1 t} \quad 0 < T \le t < T_{\max} \tag{3.28}$$

with some $b_1 > 0$. It is (3.18).

The proof of Theorem 1. Since $\partial\Omega \in C^m$, $\frac{N}{4m} < \beta < 1$, it follows from Lemma 1 that $D(A^\beta) \subset C^r(\bar\Omega)$ if $0 \le r < 2m\beta - \frac{N}{2}$. Hence, from (3.18) we obtain

$$\|u(t)\|_{C^r(\bar\Omega)} + \|v(t)\|_{C^r(\bar\Omega)} \le C_1(T)(\|u_0\|_2+\|v_0\|_2)e^{-b_1 t} \quad 0 < T \le t < T_{\max}. \tag{3.29}$$

Now we remove the assumption on $u_0, v_0 \in C_0^3(\Omega)$. Suppose that $u_0, v_0 \in L^2$ and $\|u_0\|_2^2+\|v_0\|_2^2 < d_0$. We choose $u_{0,n}, v_{0,n} \in C_0^3(\Omega)$ in such a way that $\|u_{0,n}\|_2^2+\|v_{0,n}\|_2^2 < d_0$. For any $n = 1,2,\ldots$, choose $u_{0,n}, v_{0,n} \in C_0^3(\Omega)$ and $u_{0,n} \to u_0, v_{0,n} \to v_0$ in L^2. Let $(u_n(t), v_n(t))$ be the corresponding solution of (1.1) with $u_n(0) = u_{0,n}, v_n(0) = v_{0,n}$,. From (3.29), we find that

$$\begin{aligned}\|u_n(t)\|_{C^r(\bar\Omega)} + \|v_n(t)\|_{C^r(\bar\Omega)} &\le C_1(T)(\|u_{0,n}\|_2+\|v_{0,n}\|_2)e^{-b_1 t}\\ &\le C_2(T)(\|u_0\|_2+\|v_0\|_2)e^{-b_1 t} \quad 0 < T \le t < T_{\max}.\end{aligned}$$

Here $C_1(T), C_2(T)$ are independent of n. From the above estimate, we can conclude that $u_n(t)$ and $v_n(t)$ converge as $n \to \infty$ to the functions $u(t)$ and $v(t)$ compact uniformly on $\bar\Omega \times (0, T_{\max})$ respectively. Hence $f(u_n, v_n)$ converges to $f(u,v)$ and $g(u_n, v_n)$ converges to $g(u,v)$ in $C(\bar\Omega)$ as $n \to \infty$. Moreover we find that

$$\int_0^t e^{-a_i(t-s)A}f(u_n,v_n)ds$$

and

$$\int_0^t e^{-a_i(t-s)A}g(u_n,v_n)ds$$

are uniformly integrable in L^2 with respect to n. From these facts, it is easy to see that $u(t)$ and $v(t)$ satisfy (3.19) and (3.20). Therefore, the estimates (3.2) and (3.18) must be true for $u(t)$ and $v(t)$ with $u_0, v_0 \in L^2$.

From (2.3) and (3.18), we have

$$\|u(t)\|_{W^{2,2}} + \|v(t)\|_{W^{2,2}} \le C(T)(\|u_0\|_2+\|v_0\|_2)e^{-b_1 t} \quad 0 < T \le t < T_{\max}.$$

This inequality implies $T_{\max} = +\infty$. That is to say, $(u(t), v(t))$ exists globally. The uniqueness can be proved as in [8].

The proof Theorem 1 is now completed.

4. Proof of Theorem 2

When $b = c = 0$, the problem (1.1) takes on the following form

$$\begin{cases} u_t = a \triangle u + f(u,v) & \\ v_t = d \triangle v + g(u,v) & \text{in } \Omega \times (0, +\infty) \\ u = v = 0 & \text{on } \partial\Omega \times (0, +\infty) \\ u(x,0) = u_0(x), v(x,0) = v_0(x) & \text{in } \Omega. \end{cases} \tag{4.1}$$

Suppose that nonnegative $u_0, v_0 \in C_0^3(\Omega)$ and $(u(t), v(t))$ is a nonnegative smooth solution of (4.1) for $0 \le t < T_{\max}, p > \max\{\frac{N}{4}, \frac{\alpha_0 N}{2} - 2\}$. Multiplying the first two equations in (4.1) by u^{p+1} and v^{p+1} respectively and integraing on Ω, we see that

$$\begin{aligned} &\frac{1}{p+2}\frac{d}{dt}\int_\Omega u^{p+2}dx + \frac{4a(p+1)}{(p+2)^2}\int_\Omega |\bigtriangledown u^{(p+2)/2}|^2 dx \\ = &\int_\Omega f u^{p+1} dx \le k_1 \int_\Omega (u^{p+2+\alpha_1} + u^{p+1} v^{1+\alpha_2}) dx \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &\frac{1}{p+2}\frac{d}{dt}\int_\Omega v^{p+2}dx + \frac{4d(p+1)}{(p+2)^2}\int_\Omega |\bigtriangledown v^{(p+2)/2}|^2 dx \\ = &\int_\Omega g v^{p+1} dx \le k_2 \int_\Omega (v^{p+2+\alpha_4} + u^{1+\alpha_3} v^{p+1}) dx. \end{aligned} \tag{4.3}$$

The application of Young's inequality yields

$$\begin{aligned} |u^{p+1}v^{1+\alpha_2}| &\le |u|^{p+2+\alpha_2} + |v|^{p+2+\alpha_2}, \\ |v^{p+1}u^{1+\alpha_3}| &\le |u|^{p+2+\alpha_3} + |v|^{p+2+\alpha_3}. \end{aligned} \tag{4.4}$$

Thus we obtain

$$\begin{aligned} &\frac{d}{dt}(\|u\|_{p+2}^{p+2} + \|v\|_{p+2}^{p+2}) + \frac{4(p+1)}{p+2}(a\| \bigtriangledown u^{(p+2)/2}\|_2^2 + d\| \bigtriangledown v^{(p+2)/2}\|_2^2) \\ \le\ & C\int_\Omega \sum_{i=1}^4 (|u|^{p+2+\alpha_i} + |v|^{p+2+\alpha_i}) dx \end{aligned} \tag{4.5}$$

for some $C > 0$.

Now, applying Lemma 4 and putting $k = \frac{p}{2}$, we find that

$$\int_\Omega |u|^{p+2+\alpha_i} dx \le C\|u\|_{p+2}^{(1-\theta_i)(p+2+\alpha_i)} \| \bigtriangledown u^{(p+2)/2}\|_2^{\frac{2\theta_i(p+2+\alpha_i)}{(p+2)}} \tag{4.6}$$

with $\theta_i = \dfrac{\alpha_i N}{2(p+2+\alpha_i)}$.

Since $p > \dfrac{\alpha_0 N}{2} - 2$, we see that $0 < \theta_i < 1$ and $\dfrac{2\theta_i(p+2+\alpha_i)}{p+2} = \dfrac{\alpha_i N}{p+2} < 2$. By Sobolev's inequality, we have

$$\begin{aligned}\|u\|_{p+2}^{(1-\theta_i)(p+2+\alpha_i)} &= \|u\|_{p+2}^{\alpha_i}\|u\|_{p+2}^{p+2-\frac{\alpha_i N}{2}}\\ &\le \|u\|_{p+2}^{\alpha_i}\| \nabla u^{(p+2)/2}\|_2^{2-\frac{\alpha_i N}{2}}\end{aligned}$$

Thus,

$$\int_\Omega |u|^{p+2+\alpha_i}dx \le C\|u\|_{p+2}^{\alpha_i}\| \nabla u^{(p+2)/2}\|_2^2. \tag{4.7}$$

Obviously, there is a similar estimate for v. Hence, it follows from (4.5) that

$$\begin{aligned}&\frac{d}{dt}(\|u\|_{p+2}^{p+2} + \|v\|_{p+2}^{p+2}) + \frac{4(p+1)}{p+2}(a\| \nabla u^{(p+2)/2}\|_2^2 + d\| \nabla v^{(p+2)/2}\|_2^2)\\ &\le C\sum_{i=1}^{4}(\|u\|_{p+2}^{\alpha_i} + |v|_{p+2}^{\alpha_i})(\| \nabla u^{(p+2)/2}\|_2^2 + \| \nabla v^{(p+2)/2}\|_2^2)\\ &\le C(\|u\|_{p+2}^{p+2} + |v|_{p+2}^{p+2})^{\frac{\alpha_0}{p+2}}(\| \nabla u^{(p+2)/2}\|_2^2 + \| \nabla v^{(p+2)/2}\|_2^2). \end{aligned}\tag{4.8}$$

The constant C in (4.7) and (4.8) may be different, but they are all independent of u and v.

From (4.8), we can choose a constant $\mu > 0$ such that

$$\frac{d}{dt}\varphi_3(t) \le (C\varphi_3^{\frac{\alpha_0}{p+2}}(t) - \mu)(\| \nabla u^{\frac{(p+2)}{2}}\|_2^2 + \| \nabla v^{\frac{(p+2)}{2}}\|_2^2 \quad 0 \le t < T_{\max}. \tag{4.9}$$

Here $\varphi_3(t) = \|u(t)\|_{p+2}^{p+2} + \|v(t)\|_{p+2}^{p+2}$.

After we make the assumption

$$\varphi_3(0) = (\|u_0\|_{p+2}^{p+2} + \|v_0\|_{p+2}^{p+2}) < (\frac{\mu}{C})^{\frac{p+2}{\alpha_0}} = d_1, \tag{4.10}$$

we can derive that

$$\varphi_3(t) \le \varphi_3(0) < d_1$$

and also

$$\frac{d}{dt}\varphi_3(t) + \varepsilon_2(\| \nabla u^{\frac{(p+2)}{2}}\|_2^2 + \| \nabla v^{\frac{(p+2)}{2}}\|_2^2) \le 0 \tag{4.11}$$

for some $\varepsilon_2 = \mu - [\varphi_3(0)]^{\alpha_0/(p+2)}$.

(4.11) implies (see (3.16), (3.17))

$$\varphi_3(t) \le \varphi_3(0)e^{-\varepsilon_2\lambda_1 t}, \quad 0 \le t < T_{\max}$$

with $\varepsilon_2\lambda_1 > 0$. That is

$$\|u(t)\|_{p+2}^{p+2} + \|v(t)\|_{p+2}^{p+2} \le (\|u_0\|_{p+2}^{p+2} + \|v_0\|_{p+2}^{p+2})e^{-\varepsilon_2\lambda_1 t} \tag{4.12}$$

or

$$\|u(t)\|_{p+2} + \|v(t)\|_{p+2} \le C(\|u_0\|_{p+2} + \|v_0\|_{p+2})e^{-r_2 t} \tag{4.13}$$

with $C > 0$ and $r_2 = \frac{\varepsilon_2 \lambda_1}{p+2} > 0$.

Next, we derive the estimates for $\|u(t)\|_{W^{2,p+2}}$ and $\|v(t)\|_{W^{2,p+2}}$. We know that the solution of (4.1) is equivalent to $u, v \in C_{([0,+\infty),C(\bar{\Omega}))}$ and

$$\begin{aligned} u(t) &= e^{-atA}u_0 + \int_0^t e^{-a(t-s)A} f(u,v)ds \\ v(t) &= e^{-dtA}v_0 + \int_0^t e^{-d(t-s)A} g(u,v)ds. \end{aligned} \tag{4.14}$$

Because of $p > \max\{\frac{N}{4}, \frac{\alpha_0 N}{2} - 2\}$, we can choose β such that $\max\{\frac{N}{4p}, \frac{\alpha_0 N}{2(p+2)}, \frac{1}{2}\}, < \beta < 1$. By (2.2), we have

$$\begin{aligned} \|A^\beta u(t)\|_{p+2} &\le \|A^\beta e^{-atA}u_0\|_{p+2} + \int_0^t \|A^\beta e^{-a(t-s)A} f(u,v)\|_{p+2} ds \\ &\le .Ct^{-\beta} e^{-a\delta t}\|u_0\|_{p+2} + C\int_0^t (t-s)^{-\beta} e^{-a\delta(t-s)}\|f\|_{p+2} ds \end{aligned}$$

and

$$\|A^\beta v(t)\|_{p+2} \le Ct^{-\beta} e^{-d\delta t}\|v_0\|_{p+2} + C\int_0^t (t-s)^{-\beta} e^{-d\delta(t-s)}\|g\|_{p+2} ds.$$

By assumption (H), we find

$$\begin{aligned} \|f\|_{p+2} &\le 2k_1(\|u\|^{1+\alpha_1}_{(p+2)(1+\alpha_1)} + \|v\|^{1+\alpha_2}_{(p+2)(1+\alpha_2)}) \\ \|g\|_{p+2} &\le 2k_2(\|u\|^{1+\alpha_3}_{(p+2)(1+\alpha_3)} + \|v\|^{1+\alpha_4}_{(p+2)(1+\alpha_4)}). \end{aligned} \tag{4.15}$$

By virtue of the inequality (see [7] or [8])

$$\|u\|_{(p+2)(1+\alpha_i)} \le C\|A^\beta u\|^{\theta_i}_{p+2}\|u\|^{1-\theta_i}_{p+2} \tag{4.16}$$

with $\theta_i = \frac{1}{2\beta}(\frac{1}{p+2} - \frac{1}{(p+2)(1+\alpha_i)})N = \frac{\alpha_i N}{2(p+2)(1+\alpha_i)\beta}$, as in (3.24) we can derive the following estimate

$$\|u\|^{1+\alpha_i}_{(p+2)(1+\alpha_i)} \le C\|u\|^{\alpha_i}_{p+2}\|A^\beta u\|_{p+2}. \tag{4.17}$$

Similarly,

$$\|v\|^{1+\alpha_i}_{(p+2)(1+\alpha_i)} \le C\|v\|^{\alpha_i}_{p+2}\|A^\beta v\|_{p+2}. \tag{4.18}$$

From above inequalities, we obtain

$$\begin{aligned} \|A^\beta u(t)\|_{p+2} \le\ & Ct^{-\beta} e^{-a\delta t}\|u_0\|_{p+2} \\ & + C\int_0^t (t-s)^{-\beta} e^{-a\delta(t-s)} e^{-\mu_1 s}(\|A^\beta u(s)\|_{p+2} + \|A^\beta v(s)\|_{p+2}) ds \end{aligned} \tag{4.19}$$

and

$$
\begin{aligned}
\|A^\beta v(t)\|_{p+2} \quad \le \quad & Ct^{-\beta}e^{-d\delta t}\|v_0\|_{p+2} \\
& + C\int_0^t (t-s)^{-\beta}e^{-d\delta(t-s)}e^{-\mu_2 s}(\|A^\beta u(s)\|_{p+2} + \|A^\beta v(s)\|_{p+2})ds
\end{aligned} \tag{4.20}
$$

with positive constants μ_1 and μ_2.

Let

$$\varphi_4(t) = \|A^\beta u(t)\|_{p+2} + \|A^\beta v(t)\|_{p+2}$$

and

$$D = \min\{a\delta, d\delta\}, E = \min\{\mu_1, \mu_2\}.$$

Then, it follows from (4.19) and (4.20) that

$$\varphi_4(t) \le C(\|u_0\|_{p+2} + \|v_0\|_{p+2})t^{-\beta}e^{-Dt} + C\int_0^t (t-s)^{-\beta}e^{-D(t-s)}e^{-Es}\varphi_4(s)ds.$$

This inequality implies, $\exists\ b_0 > 0$, such that for any $0 < T < T_{\max}$

$$
\begin{aligned}
\varphi_4(t) \quad = \quad & \|A^\beta u(t)\|_{p+2} + \|A^\beta v(t)\|_{p+2} \\
\le \quad & C(T)(\|u_0\|_{p+2} + \|v_0\|_{p+2})e^{-b_2 t}, \quad 0 < T \le t < T_{\max}.
\end{aligned} \tag{4.21}
$$

Note that $\beta > \dfrac{1}{2}$. We use (2.3) and obtain

$$
\begin{aligned}
& \|u(t)\|_{W^{2,p+2}} + \|v(t)\|_{W^{2,p+2}} \\
& \le C(T)(\|u_0\|_{p+2} + \|v_0\|_{p+2})e^{-b_2 t} \quad 0 < T \le t < T_{\max},
\end{aligned} \tag{4.22}
$$

and

$$\|u(t)\|_{C^r(\bar\Omega)} + \|v(t)\|_{C^r(\bar\Omega)} \le C(T)(\|u_0\|_{p+2} + \|v_0\|_{p+2})e^{-b_2 t} \quad 0 < T \le t < T_{\max} \tag{4.23}$$

with $0 < r < 4\beta - \dfrac{N}{p+2}$.

From (4.22), we see that $T_{\max} = +\infty$. This means that (u, v) is a global solution of (4.1). We note that the estimates (4.22) and (4.23) are made under the smoothness assumption on u_0 and v_0. We can remove this smoothness as in the proof of Theorem 1. The uniqueness of the solution follows immediately from the Lipschitz continuity of f and g. The proof of Theorem 2 is completed.

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一类强藕合抛物型方程组整体解的存在性及渐近性态

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摘　要

本文把半群理论与能量方法结合起来，证明了一类强藕合非线性抛物型方程组的初边值问题解的整体存在性，并给出了解的$W^{2,p}$全局估计.