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Stability of Solution of Oblique Derivative Problem for Nonlinear Complex Elliptic Equation of First Order *

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Abstract In this paper we discuss the stability of the solution of the oblique derivative problem for the nonlinear complex elliptic equation of first order. This result is derived from the prior estimate for the solution of the relative boundary value problems.

We already discussed the stability of the solution of the obilque derivative problem for the system of second-order nonlinear elliptic equations (See [1],[2]). In this paper we shall deal with the stability of the solution of the oblique derivative problem for the nonlinear complex elliptic equation of first-order by using the method of prior estimates for solutions of boundary value problems.

1. Conditions for a kind of complex equations

Suppose that D is an $(N+1)$ connected $(0 \le N < \infty)$ and bounded region in z−plane with boundary $\Gamma \in C^2_\mu (0 < \mu < 1)$. Without loss of generality, we assume that D is an $(N+1)$-connected and bounded circular region in the interior of the unit circle, $0 \in D$ and its boundary $\Gamma = \bigcup_{j=0}^{N} \Gamma_j$, where $\Gamma_j = \{z : |z - z_j| = \gamma_j\} (j = 1, \cdots, N)$ is in the interior of $\Gamma_0 \equiv \Gamma_{N+1} = \{z : |z| = 1\}$, which can be always attained by using a conformal mapping with respect to z (See [3]).

Now we consider the following uniformly first-order nonlinear complex elliptic equation in z-plane:

$$\begin{cases} w_{\bar z} = F(z, w, w_z), \\ F = Q_1 w_z + Q_2 \overline{w}_{\bar z} + A_1 w + A_2 \bar w + A_3, \ A_j = A_j(z, w), \ j = 1, 2, 3, \ z \in D, \end{cases} \tag{1.1}$$

where $Q_j = Q_j(z, w, w_z)$ $(j = 1, 2)$.

The oblique derivative problem for the complex equation (1.1) on D is

Problem P To find a continuously differentiable solution $w(z)$ of (1.1) on D such that it satisfies the boundary condition:

$$\mathrm{Re}[\overline{\lambda(z)} w_z + s(z) w(z)] = \tau(z), \quad z \in \Gamma, w(1) = 0 \tag{1.2}$$

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and the point-type condition:

$$\begin{array}{c} \mathrm{Im}[\overline{\lambda(a_j)}w_z(a_j)+s(a_j)w(a_j)]=b_j, \\ j\in\{j\}=\begin{cases} 0,1,\cdots,2K-N, & K\geq N, \\ 0,N-K+1,\cdots,N, & 0\leq K<N, \end{cases} \end{array} \tag{1.3}$$

where $K=\frac{1}{2\pi}\Delta_\Gamma \arg\lambda(z)$ is the index of the problem,$a_j\in\Gamma_j(j=1,\cdots N),a_j\in\Gamma_0(j=0,N+1,\cdots,2K-N)$ and $b_j(j=0,1,\cdots,2K-N)$ are real constants,$|\lambda(z)|=1$,moreover

$$C_\alpha[\lambda(z),\Gamma]\leq d,\quad C_\alpha[s(z),\Gamma]\leq\epsilon d,\quad \tau(z)\in C_\alpha(\Gamma), \tag{1.4}$$

where $\alpha(1/2<\alpha<1),d(0<d<\infty)$ and $\epsilon(0<\epsilon<1)$ are all real constants. For convenience, we suppose $a_0=1,\lambda(1)=1,\tau(1)=0$ and $b_0=0$.

Now we suppose that the complex equation(1) satisfies

Conditions C: 1) Function $F(z,w,v)$ is continuous with respect to $z\in\bar{D},w\in E$ and $v\in E$ (E is the complex plane).

2) Its generalized derivatives of first order exist for $z(\in\bar{D}),\bar{z},w(\in E),\bar{w},v(\in E)$ and $\bar{v}$; they are measurable with respect to $z\in D$ and to arbitrary continuous functions w,v in the region D and continuous with respect to $w\in E,v\in E$ for almost all $z\in D$, as well as satisity the following inequalities:

$$\left.\begin{array}{l} \|F_z\|_{L_p(\bar{D})}\leq\frac{1-q_0}{2}\|A_3(z,w)\|_{L p_0(\bar{D})}\leq k_0, \\ \|F_{\bar{w}}\|_{L_p(\bar{D})}\leq\epsilon(1-q_0)\|A_3(z,w)\|_{L p_0(\bar{D})}, \\ \|F_w\|_{L_p(\bar{D})}\leq k_0, \end{array}\right\} \tag{1.5}$$

where $\epsilon(0<\epsilon<1)$ and $p_0(2<p_0<p<\infty)$ are constants.

3) $F(z,w,\bar{w})$ and $F_v,F_{\bar{v}}$ satisfy

$$|F(z_1,w_1,v_1)-F(z_2,w_2,v_2)|\leq k_0|z_1-z_2|+\epsilon k_0|w_1-w_2|+q_0|v_1-v_2|,\ z_1,z_2\in\Gamma, \tag{1.6}$$

$$|F(1,w(1),v(1))|\leq k_0, \tag{1.7}$$

$$|F_v|+|F_{\bar{v}}|\leq q_0<1, \tag{1.8}$$

where $k_0(0<k_0<\infty)$ and q_0 are all constants.

2. Representation and estimate for solution of boundary value problem

Let $w(z)$ be the solution for the complex equation (1.1) satisfying condition C with $w(z)\in W^2p_0,2<p_0<0$. Using the same method as [4], we can obtain the representation

$$\left.\begin{array}{ll} w(z)=X(z)+H\rho+Y(z), & X(z)=\int_1^z\Phi_1(z)dz+\overline{\int_1^z\Phi_2(z)dz}, \\ Y(z)=\sum_{m=1}^{N}\int_1^z\frac{d_m}{z-z_m}dz, & H\rho=\frac{2}{\pi}\int\int_D\ln|\frac{\xi-z}{\xi-1}|\rho(\xi)dG_\xi,\ \ \rho(z)=w_{\bar{z}z} \end{array}\right\} \tag{2.1}$$

where complex constants $d_m (m = 1, \cdots, N)$ are chosen in such a way that makes $w(z)$ be monotropic in D, $\Phi_1(z)$ and $\Phi_2(z)$ are analytic functions in D, and $\Phi_2(z)$ satisfies the boundary condition

$$\begin{cases} \mathrm{Re}[\Phi(z) + TV_{\bar z}] & = \mathrm{Re}\, F(z, w, V) + H(z), \ z \in \Gamma, \\ \Phi(1) + TV_{\bar z}|_{z=1} & = F(1, w(1), V(1)). \end{cases}$$

$$H(z) = \begin{cases} 0, & z \in \Gamma_0, \\ H_j, & z \in \Gamma_j, \end{cases} \quad (j = 1, \cdots, N),$$

where $w(z)$ is a continuously differentiable function of the closed region $\bar{D}$ and $H_j's$ are undetermined constants,$\Phi(z)$ is an analytic function in D and continuous on the closed region $\bar{D}$, and T is the integral operator defined by $T\rho = -\frac{1}{\pi} \int\int_D \frac{\rho(\xi)}{\xi - z} dG_\xi$, $\rho(z) = V_{\bar z}$.

Now we give the prior estimate for the solution of the boundary value problem. We assume that

$$L_p[A_3(z, w), \bar{D}] \le k_1, \quad C_\alpha[r(z), \Gamma] \le k_2, \quad \max_{j \in \{j\}} |b_j| \le k_3. \tag{2.2}$$

Theorem 1 *Suppose that the complex equation (1.1) satisfies Conditions C, $w(z) \in W^2 p_0(\bar{D})(2 < p_0 < p)$ and $\epsilon's$ in (1.4)-(1.6) are sufficiently small. Then for any solution $w(z)$ of the from (2.1) satisfying*

$$C^1[w(z), \bar{D}] \le M_0 (M_0 \textit{is an undetermined constant}), \quad w(1) = 0, \tag{2.3}$$

the following estimate holds

$$S(w) = C^1_\beta[w(z), \bar{D}] + Lp_0[|w_{zz}| + |w_{\bar z z}|, \bar{D} \le M_1(k_1 + k_2 + k_3), \tag{2.4}$$

where $\beta = \min(\alpha, \frac{p_0 - 2}{p_0})$ $(2 < p_0 < p), M_1 = M_1(q_0, p_0, k_0, D, \alpha, K, d)$.

Proof Set $V = w_z$ and change the complex equation (1.1) into

$$V_{\bar z} = q_1^* V_z + q_2^* \bar{V}_{\bar z} + B_1^* V + B_2^* \bar{V} + B_3^*, \tag{2.5}$$

where $q_1^* = q_1/q_3, q_2^* = \bar{q}_1 q_2/q_3, B_1^* = B_1/q_3, B_2^* = \bar{B}_1 q_2/q_3, B_3^* = (B + \bar{B} q_3)/q_3, q_3 = 1 - |q_2|^2$ and $|q_1^*| + |q_2^*| \le q_0^* < 1$.

Now we claim that v satisfies the equation:

$$v_{\bar z} = q_1^* v_z + q_2^* \bar{v}_{\bar z} + B_1^* v + B_2^* \bar{v} + B_3^*/k, \tag{2.6}$$

under the boundary condition:

$$\mathrm{Re}[\overline{\lambda(z)} v(z)] = r(z)/k - \mathrm{Re}[s(z) w(z)/k] + h(z), \quad z \in \Gamma, \quad w(1)/k = 0, \tag{2.7}$$

and the point-type condition:

$$\mathrm{Im}[\overline{\lambda(a_j)} v(a_j) + s(a_j) w(a_j)/k] = b_j/k \quad j \in \{j\}, \tag{2.8}$$

provided $k = k_1 + k_2 + k_3$ and $v = w_z/k$. By §3-§6,Ch5 in [4],there is a unique solution of the problem (2.6)-(2.8). It is also obvious that $C_\alpha[r(z)/k, \Gamma] \le 1$ and $\max_{j \in \{j\}} |b_j/k| \le$

1. Then, due to Conditions C and (2.3), no matter how large the constant M_0 is, it always holds for B_3^* in (2.5) that $\|B_3^*\|_{Lp_0(D)} \le \|A_3(z,w)\|_{L_p(\bar{D})}$, provided that ϵ in (1.5) is sufficiently small. Hence $\|B_3^*/k\|_{Lp_0(\bar{D})} \le 1$. By Theorem 5.6, Ch5 in [4], we obtain the following estimate:

$$C_\beta[v(z),\bar{D}] + Lp_0[|v_z| + |v_{\bar{z}}|, \bar{D}] \le M_2 = M_2(q_0, p_0, k_0, D, K, \alpha, d), \tag{2.9}$$

where $\beta = \min(\alpha, \frac{p_0-2}{p_0}), 2 < p_0 < p$.

By (2.3) and (2.9), as long as ϵ in (1.6) is sufficiently small, the following estimate will hold:

$$|F(z_1, w(z_1), kv(z_1)) - F(z_2, w(z_2), kv(z_2))| \le (2k_0 + M_2)|z_1 - z_2|^\beta, \quad z_1, z_2 \in \Gamma. \tag{2.10}$$

This implies that $\Phi_2(z)$ in (2.1) satisfies

$$C_\beta[\Phi_2(z)/k, \bar{D}] \le M_3 = M_3(q_0, p_0, k_0, D, K, \alpha, d). \tag{2.11}$$

Combining (2.9),(2.11) and the fact that $(H\rho)_z = T\rho$, we get the estimate for $w(z)/k$:

$$C_\beta[w(z)/k, \bar{D}] + Lp_0[|w_{zz}/k| + |w_{\bar{z}z}/k|, \bar{D}] \le M_1, \quad z \in \bar{D}, \tag{2.12}$$

that derives the estimate (2.4) and $w(1) = 0$. Meanwhile,(2.3) also holds if $M_0 = kM_1$. □

3. Stability of solution of boundary value problem

In order to dicuss the stability of the solution of Porblem P, we assume that the complex equation:

$$\begin{cases} w_{\bar{z}} = & F^m(z, w, w_z), \quad F^m = Q_1^m w_z + Q_2^m \bar{w}_{\bar{z}} + A_1^m w + A_2^m \bar{w} + A_3^m, \\ Q_j^m = & Q_j^m(z, w, w_z), \ j = 1, 2, \quad A_j^m = A_j^m(z, w), j = 1, 2, 3, m = 1, 2 \end{cases} \tag{3.1}$$

satisfies not only Conditions C, but also

Conditions C* For any $w_1(z), w_2(z) \in C_\beta^1(\bar{D})$ and $v(z) \in C_\beta(\bar{D})$, the following inequatlities hold:

$$\begin{aligned} & L_\infty[Q_j^1(z, w_1, v) - Q_j^2(z, w_2, v), D] \\ \le\ & L_\infty[Q_j^1(z) - Q_j^2(z), D] + \epsilon C_\beta[w_1 - w_2, \bar{D}], \quad j = 1, 2. \end{aligned} \tag{3.2}$$

and

$$Lp_0[A_j^1(z, w_1) - A_j^2(z, w_2), \bar{D}] \le Lp_0[A_j^1(z) - A_j^2(z), \bar{D}] + \epsilon C_\beta[w_1 - w_2, \bar{D}], \quad j = 1, 2, 3, \tag{3.3}$$

where $Q_j^m(z) = Q_j^m(z, 0, 0), j = 1, 2, \ A_j^m(z) = A_j^m(z, 0), j = 1, 2, 3, m = 1, 2$.

Then the boundary conditions in Problem P for the complex equation (3.1) turn to

$$\mathrm{Re}[\overline{\lambda^m(z)} w_{mz} + s^m(z) w_m(z)] = r^m(z), \quad z \in \Gamma, w_m(1) = 0, m = 1, 2 \tag{3.4}$$

and

$$\mathrm{Im}[\overline{\lambda^m(a_j)}w_{mz}(z_j)+s^m(a_j)w_m(a_j)]=b_j^m \quad j\in\{j\}, m=1,2, \tag{3.5}$$

where $\lambda^m(z), r^m(z)$, and $s^m(z)$ satisfy (1.4), a_j and b_j^m are similar to those a_j and b_j in (1.3), but here provided

$$L_p[A_3^m(z,w),\bar{D}]\le\frac{2k_0}{1-q_0},\quad C_\alpha[r^m(z),\Gamma]\le d,\quad \max_{j\in\{j\}}|b_j^m|\le d,\quad m=1,2. \tag{3.6}$$

According to the results in §1 and §2, we obtain the following stability theorem for the solution of the boundary value problem:

Theorem 2 *Suppose that the complex equation (3.1) satifies Conditions C and Conditions C*; ϵ's in (1.4), in Condition C and Conditions C* are all sufficiently small. Then the solution of Problem P for the complex equation (3.1), say $w_m(z)(m=1,2)$, has the following properties:*

$$\begin{aligned} s(w_1-w_2) &= C_\beta^1[w_1-w_2,\bar{D}]+Lp_0[|w_1-w_2)_{zz}|+|(w_1-w_2)_{\bar{z}z}|,\bar{D}] \\ &\le M_4\{\sum_{j=1}^{2}L_\infty[Q_j^1(z)-Q_j^2(z),D]+\sum_{j=1}^{3}Lp_0[A_j^1(z)-A_j^2(z),\bar{D}] \\ &+C_\beta[\lambda^1(z)-\lambda^2(z),\Gamma]+C_\beta[r^1(z)-r^2(z),\Gamma] \\ &+C_\beta[s^1(z)-s^2(z),\Gamma]+B\} \end{aligned} \tag{3.7}$$

where $B=\max_{j\in\{j\}}|b_j^1-b_j^2|, \beta=\min(\alpha,1-2/p_0), 2<p_0<p, M_4=M_4(q_0,p_0,k_0,D,k,\alpha,d)$.

Proof Since $w_m(z)(m=1,2)$ is the solution of Problem P for the complex equation (3.1), $w(z)=w_1(z)-w_2(z)$ is surely a continulusly differentiable solution of the following first-order equation on $\bar{D}$:

$$w_{\bar{z}}=Q_1^1(z,w_1,w_{1z})w_z+Q_2^1(z,w_1,w_{1z})\bar{w}_{\bar{z}}+A_2^1(z,w_1)w+A_2^1(z,w_1)\bar{w}+A(z), \tag{3.8}$$

$$\begin{aligned} A(z) \;&=\; |Q_1^1(z,w_1,w_{2z})-Q_1^2(z,w_2,w_{2z})]w_{2z}+|Q_2^1(z,w_1,w_{2z})-Q_2^2(z,w_2,w_{2z})]\bar{w}_{2\bar{z}} \\ &+\; [A_1^1(z,w_1)-A_1^2(z,w_2)]w_2+[A_2^1(z,w_1)-A_2^2(z,w_2)]\bar{w}_2+A_3^1(z,w_1)-A_3^2(z,w_2), \end{aligned}$$

where

$$\begin{aligned} Lp_0[A(z),\bar{D}] \;&\le\{\sum_{j=1}^{2}L_\infty[Q_j^1(z)-Q_j^2(z),D]+2\epsilon C_\beta[w_1-w_2,D]\}C_\beta[w_2,\bar{D}] \\ &+\{\sum_{j=1}^{3}Lp_0|A_j^1(z)-A_j^2(z),\bar{D}]+3\epsilon C_\beta[w_1-w_2,\bar{D}]\}+1] \end{aligned} \tag{3.9}$$

and $w(z)$ satisfies the following boundary codition:

$$\mathrm{Re}[\overline{\lambda^1(z)}w_z+s^1(z)w(z)]=R(z),\quad z\in\Gamma, w(1)=0. \tag{3.10}$$

$$R(z)=r^1(z)-r^2(z)-\mathrm{Re}[\overline{\lambda^1(z)-\lambda^2(z))}w_{2z}+(s^1(z)-s^2(z))w_2(z)],\quad z\in\Gamma$$

and point-type condition:

$$\mathrm{Im}[\overline{\lambda^1(a_j)}w_z(z_j) + s^1(a_j)w(a_j)] = B_j, \quad j \in \{j\}. \tag{3.11}$$

$$B_j = b_j^1 - b_j^2 - \mathrm{Im}[\overline{(\lambda^1(a_j) - \lambda^2(a_j))}w_{2z}(a_j) + (s^1(a_j) - s^2(a_j))w_2(a_j)],$$

where

$$C_\beta[R(z), \Gamma] \le C_\beta[r^1 - r^2, \Gamma] + C_\beta[\lambda^1 - \lambda^2, \Gamma]C_\beta[w_{2z}, \bar{D}] + C_\beta[s^1 - s^2, \Gamma]C_\beta[w_2, \bar{D}] \tag{3.12}$$

and

$$|B_j| \le |b_j^1 - b_j^2| + C_\beta[\lambda^1 - \lambda^2, \Gamma]C_\beta[w_2, \bar{D}] + C_\beta[s^1 - s^2, \Gamma]C_\beta[w_2, \bar{D}], \quad j \in \{j\}. \tag{3.13}$$

By Theorem 1, we have $s(w_j) = C_\beta^1[w_j, \bar{D}] + Lp_0[|w_{jzz}| + |w_{j\bar{z}z}|, \bar{D}] \le M_5$, $j = 1, 2$, where $M_5 = M_5(q_0, p_0, k_0, D, K, \alpha, d)$.

Hence,

$$\begin{aligned}
s(w_1 - w_2) &= s(w) = C_\beta^1[w, \bar{D}] + Lp_0[|w_{zz}| + |w_{\bar{z}z}|, \bar{D}] \\
&\le M_6\{Lp_0[A(z), \bar{D}] + C_\beta[R(z), \Gamma] + |B_j|\} \\
&\le \{[\sum_{j=1}^{3} L_\infty(Q_j^1(z) - Q_j^2(z), D) + 2\epsilon C_\beta(w, \bar{D})]C_\beta(w_{2z}, \bar{D}) \\
&+[\sum_{j=1}^{3} Lp_0(A_j^1(z) - A_j^2(z), \bar{D}) + 3\epsilon C_\beta(w_1 - w_2, \bar{D})][C_\beta(w_2, \bar{D}) + 1] \\
&+C_\beta(r^1 - r^2, \Gamma) + 2C_\beta(\lambda^1 - \lambda^2, \Gamma)C_\beta(w_{2z}, \bar{D}) \\
&+2C_\beta(s^1 - s^2, \Gamma)C_\beta(w_2, \bar{D}) + B\} \\
&\le M_6\{[\sum_{j=1}^{2} L_\infty(Q_j^1(z) - Q_j^2(z), D) + 2\epsilon C_\beta(w, \bar{D})]M_5 \\
&+[\sum_{j=1}^{3} Lp_0[A_j^1(z) - A_j^2(z), \bar{D}] + 3\epsilon C_\beta(w, \bar{D})](M_5 + 1) \\
&+C_\beta(r^1 - r^2, \Gamma) + 2C_\beta(\lambda^1 - \lambda^2, \Gamma)M_5 + 2C_\beta(s^1 - s^2, \Gamma)M_5 + B\},
\end{aligned} \tag{3.14}$$

where $M_6 = M_6(q_0, p_0, k_0, D, K, \alpha, d)$.

It is obvious that (3.14) implies (3.7) provided ϵ is so sufficiently small that $1 - 5\epsilon M_6(M_5 + 1) \ge 1/2$. This result shows the stability of the solution of Problem P for the complex equation (3.1). □

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