the Lemma 2.2 of [5], $P_r(u) = P_r(v) = 1$. By Theorem 2.3 of [5], $0 \in P_{P_r}(x)$. Since Y is a P_r-homogeneous imbedded subspace, we have $P_r(x+u) = P_r(x+v)$. We may assume that $\lambda = P_r(x+u) > 0$. By Lemma 2.1, we have

$$P_r(x+u) = \lambda P_{\psi(\lambda)r}(x+u) = P_r(x+v) = \lambda P_{\psi(\lambda)r}(x+v).$$

So $P_{\psi(\lambda)r}(x+u) = \lambda P_{\psi(\lambda)r}(x+v) = 1$. By Lemma 2.1, one has $f(x+u) = \psi(\lambda)r = f(x+v)$. So Y is an f-homogeneous imbedded subspace of X. □

Remark By Theorem 3.3, if the condition (F2) is replaced by (F1) in Lemma 2.3, 2.4 and Theorem 3.1 and 3.2, we have the same conclusions respectively.

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局部凸空间中的齐次嵌入子空间

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在本文中, 将齐次嵌入子空间概念引入了局部凸空间中, 并讨论了它们了逼近性质.

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f-homogeneous Imbedded Subspaces in Locally Convex Spaces

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Abstract. In this paper, we introduce the concept of the homogeneous imbedded subspace in locally convex spaces and study the approximation properties of these subspaces.

Key words: f-homogeneous imbedded subspace, f-proximinal subspace

1. Introduction

Let X and X' be a pair of linear spaces put in duality by a bilinear form $\langle,\rangle$. We assume that this bilinear form $\langle,\rangle$ is separating, i.e., for each $x \in X$ and $x \neq 0$, there exists y in X' such that $\langle x, y\rangle \neq 0$ and, for each $y \in X'$ and $y \neq 0$, there exists an $x \in X$ such that $\langle x, y\rangle \neq 0$. A topology on X is said to be ***compatible*** if it is a separated locally convex topology for which continuous linear functions on X are precisely of the form

$$\langle\,\cdot\,, y\rangle : x \to \langle x, y\rangle, \quad for \quad y \in X'.$$

Let f be a continuous convex function defined on X and satisfying $f(0) = 0$. Given a nonempty Y of X and $x \in X$, let

$$\begin{aligned} f_Y(x) &= \inf\{f(x-y);\ \ y \in Y\}; \\ P_f(x) &= \{y \in Y;\ \ f_Y(x) = f(x-y)\}; \end{aligned}$$

The set-valued mapping P_f is called f-metric projection supported on Y. Y is said to be f-proximinal (resp. f-Chebyshev) if $P_f(x)$ is nonempty (resp. $P_f(x)$ is a singleton) for each $x \in X$.

For $r > 0$, let $S_r = \{x \in X;\ \ f(x) \leq r\}$ denote the sub-level subset of f, and $P_r(x) = \inf\{\lambda > 0;\ \ x \in \lambda S_r\}$ denote the Minkowski gauge of S_r. Then P_r is a non-negative continuous sublinear function.

In section 2, we obtain the element properties of homogeneous imbedded subspace.

In section 3, we investigate the f-approximation with respect to a homogeneous imbeded subspace and the f-Chebyshev subspaces.

Let X be locally convex and f a real function defined on X. Consider the conditions:

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(F1) There exists a continuous bijection $\psi : R_+ \mapsto R_+$ such that, for any $x \in X$ and $\lambda \geq 0$, $f(\lambda x) = \psi(\lambda) f(x)$ and f is continuous and convex.

(F2) f is a symmetric sublinear function.

Obviously, if there exists an $x \in X \backslash \{0\}$ such that $f(x) > 0$, then ψ is a convex function and $\psi(t) \to \infty$ as $t \to \infty$.

2. Preliminary and notations

Lemma 2.1 *(D.V.Pai and P.Govindarajulu [8]) Suppose f satisfies the condition (F1) and $0 = f(0) \leq f(x)$. Then for any $\alpha, \beta > 0$,*

$$S_\alpha = (1/\beta) S_{\psi(\beta)\alpha}, \quad P_{\alpha,Y} = \beta P_{\psi(\beta)\alpha,Y}.$$

By this Lemma, we have $P_{P_{\alpha},Y}(x) = P_{P_{\beta},Y}(x)$, for any $x \in X$ and $\alpha, \beta > 0$.

If X is a normed linear space and f is the norm on X, then the following definition is as that in [1].

Definition 2.2 *Let X be a locally convex space and f a real function defined on X. Y is an f-proximinal subspace of X. Y is called to be an f-homogeneous imbedded subspace of X if, for any $x \in X$ and $u, v \in Y$, $0 \in P_f(x)$ and $f(u) = f(v)$ imply that $f(x+u) = f(x+v)$.*

Firstly, we consider some properties of homogeneous imbedded subspace of X.

Lemma 2.3 *Let f be real function defined on X which satisfies the condition (F2) and Y an f-homogeneous imbedded subspace of X. Given $x \in X, u \in P_f(x)$ and $w \in Y$, if $f(x-y) < f(x-w)$, then for every $y \in Y$, we have $f(u-y) < f(u-w)$.*

Proof Assume that the conclusion is fales. then there exists a $y \in Y$ such that $f(u-y) \geq f(u-w)$.

Case 1 $f(u-y) = f(u-w)$. *Since $u \in P_f(x)$, we have $0 \in P_f(x-u)$. Since Y is an f-homogeneous imbedded subspace of X and $u, w, y \in Y$, we have*

$$\begin{aligned} f(x-y) &= f[(x-u)+(u-y)] \\ &= f[(x-u)+(u-w)] \\ &= f(x-w). \end{aligned}$$

This is in contradiction with the assumption.

Case 2 $f(u-y) > f(u-w)$. *If $f(u-w) = 0$, then*

$$\begin{aligned} f(x-w) &= f[(x-u)+(u-w)] \\ &= f(x-u)+f(u-w) \\ &= f_Y(x). \end{aligned}$$

Since $w \in Y$, $w \in P_f(x)$. This is impossible since $y \in Y$ and $f(x-y) < f(x-w)$.

Now, we assume that $f(u-y) > f(u-w) > 0$. Then there exists a $0 < t_0 < 1$ such that

$$f(u-w) = t_0 f(u-y) = f[t_0(u-y)].$$

Since Y is an f-homogeneous imbedded subspace of X and $u \in P_f(x)$, we have $f(x-u) \leq f(x-w)$. Hence we have

$$\begin{aligned} f(x-w) &= f[(x-u)+(u-w)] \\ &= f[(x-u)+t_0(u-y)] \\ &= f[t_0(x-y)+(1-t_0)(x-u)] \\ &\leq t_0 f(x-y)+(1-t_0)f(x-u) \\ &\leq t_0 f(x-y)+(1-t_0)f(x-w). \end{aligned}$$

It implies that $t_0 f(x-w) \leq t_0 f(x-w)$. This is impossible since $t_0 > 0$ and $f(x-y) < f(x-w)$. □

Lemma 2.4 *Assume that X, f and Y satisfy the conditions of Lemma 2.3. Given $x \in X$, then $0 \in P_f(x)$ if and only if, for any $u, v \in Y$, $f(u) = f(v)$ implies $f(x+u) = f(x+v)$.*

Proof The necessiarity is the definition of homogeneous imbedded subspace of X.

Since Y is f-proximinal, so $P_f(x) \neq \emptyset$. Let $u \in P_f(x)$. Since f is symmetric, we have that $f(u) = f(-u)$. By definition, we have $f(x+u) = f(x-u)$. Since $f(x+u) = f_Y(x)$, $-u \in P_f(x)$. Hence

$$f(x) = f(x + \frac{u-u}{2}) \leq [f(x+u)+f(x-u)]/2 = f_Y(x).$$

Thus $0 \in P_f(x)$. □

Lemma 2.5 *Let X be a locally convex space, f a real function defined on X which satisfies the condition (F2) and Y an f-proximinal subspace of X. Suppose that for $x \in X$ and $u, v \in Y$, when $f(u) = f(v) = 1$ and $0 \in P_f(x)$, one has $f(x+u) = f(x+v)$. Then Y is a homogeneous imbedded subspace of X.*

Proof Let $x \in X$ such that $0 \in P_f(x)$ and $u, v \in Y$ such that $f(u) = f(v)$. If $f(u) = f(v) = 0$, then

$$f(x+u) \leq f(x) + f(u) = f(x), \ \textit{and} \ \ f(x) = f(x+u-u) \leq f(x+u).$$

Hence $f(x) = f(x+u)$. Similarly, $f(x) = f(x+v)$. So $f(x+u) = f(x+v)$.

Now, we assume that $f(u) = f(v) = r > 0$. Then $f(u/r) = f(v/r) = 1$. obviously, $0 \in P_f(x/r)$. By assumption, we have

$$\begin{aligned} f(x-u)/r &= f(\frac{x}{r} - \frac{u}{r}) \\ &= f(\frac{x}{r} - \frac{v}{r}) = f(x-v)/r. \end{aligned}$$

So we have $f(x+u) = f(x+v)$ and Y is a homogeneous imbedded subspace of X. □

3. The Main Theorems

Theorem 3.1 *Let X be a locally convex space, f a real function defined on X which satisfies the condition (F2) and Y an f-homogeneous imbedded subspace of X. Then Y is f-Chebyshev subspace of X if and only if $f(y) > 0$ when $y \in Y \setminus \{0\}$, that is, the restriction $f|_Y$ of f on Y is a norm on Y.*

Proof Let Y be an f-Chebyshev subspace of X. Suppose that there exists a $y_0 \in Y \setminus \{0\}$ such that $f(y_0) = 0$. Let $x \in X \setminus Y$. Since Y is f-proximinal, $P_f(x) \neq \emptyset$. Let $u \in P_f(x)$. Let $x_0 = x - u$. Then $0 \in P_f(x_0)$. Hence $f(x_0) = f_Y(x_0)$ and

$$f(x_0 + y_0) \leq f(x_0) + f(y_0) = f(x_0) = f_Y(x_0).$$

This implies that $-y_0 \in P_f(x_0)$. This is in contradiction with Y being an f-Chebyshev subspace since $0,\ -y_0 \in P_f(x_0)$ and $y_0 \neq 0$. We complete the proof of necessarity.

To show the sufficience, suppose that Y is not an f-Chebyshev subspace. Then there exist $x \in X$ and $y_1,\ y_2 \in P_f(x)$ such that $y_1 \neq y_2$. Let $x_0 = x - y_1$ and $y_0 = y_2 - y_1$. Then $y_0 \neq 0$ and $0,\ y_0 \in P_f(x_0)$. Thus we have

$$f(x_0) = f_Y(x_0) = f(x_0 - y_0).$$

Since f is symmetric, we have $f(y_0) = f(-y_0)$. Since $0 \in P_f(x_0 - y_0)$ and Y is f-homogeneous imbedded subspace of X, we have

$$f_Y(x_0) = f(x_0) = f[(x_0 - y_0) + y_0] = f(x_0 - 2y_0).$$

So $2y_0 \in P_f(x)$. If $ky_0 \in P_f(x_0)$ for $k = 1, 2, \cdots, n$ where $n \geq 3$, then $ny_0,\ (n-1)y_0 \in P_f(x_0)$. Hence $0 \in P_f(x_0 - ny_0)$. $f(y_0) = f(-y_0)$ implies that

$$\begin{aligned} f_Y(x_0) &= f[x_0 - (n-1)y_0] \\ &= f((x_0 - ny_0) + y_0] \\ &= f[(x_0 - ny_0) + (-y_0)] \\ &= f[x_0 - (n+1)y_0]. \end{aligned}$$

Therefor, $(n+1)y_0 \in P_f(x_0)$. By induction, we have $ny_0 \in P_f(x_0)$ for every integer n. Since

$$\begin{aligned} nf(y_0) &= f(ny_0) = f[(-x_0 + ny_0) + x_0] \\ &\leq f(x_0) + f(x - ny_0) \\ &= 2f_Y(x_0), \end{aligned}$$

we have $0 \leq f(y_0) \leq 2f_Y(x_0)/n \to 0$ when $n \to \infty$. So $f(y_0) = 0$. This is in contradiction with $y_0 \neq 0$ and the assumption. □

Theorem 3.2 *Let X and f satisfy the conditions of Theorem 3.1 and Y an f-proximinal and closed subspace of X. Then Y is an f-homogeneous imbedded subspace of X if and* *only if, given $x \in X$, $y \in P_f(x)$ satisfing $f(y) \neq 0$, for every $u \in Y$, when $f(u) = 1$, one has*

$$f(x - \frac{y}{f(y)}) \leq f(x - u).$$

Proof ($\Rightarrow$). Assume that there exists a $u_0 \in Y$ such that $f(u_0) = 1$ and $f(x - u_0) < f(x - \frac{y}{f(y)})$. Since u_0, $y/f(y) \in Y$ and $y \in P_f(x)$, by Lemma 2.2, we have

$$\begin{aligned} f(y - u_0) &< f(y - \frac{y}{f(y)}) \\ &= |1 - \frac{1}{f(y)}| f(y) \\ &= |f(y) - 1|. \end{aligned} \tag{1}$$

Since $f(u_0) = 1$, we have

$$f(y) = f[(y - u_0) + u_0] \leq f(y - u_0) + f(u_0) \leq f(y - u_0) + 1. \tag{2}$$

Similarly, we have

$$1 = f[(u_0 - y) + y] \leq f(u_0 - y) + f(y) = f(y - u_0) + f(y). \tag{3}$$

By (2) and (3), we have $|f(y) - 1| \leq f(y - u_0)$. This is in contradiction with (1).

($\Leftarrow$). Assume that Y is not a homogeneous imbedded subspace of X.

By Lemma 2.4, there exist $x \in X$ and $u, v \in Y$ such that $0 \in P_f(x)$, $f(u) = f(v) = 1$ and $f(x+u) \neq f(x+v)$. Without loss of generality, we may assume that $f(x+v) < f(x+u)$. Since $\lim_{t \mapsto 0} f(v + tu) = f(v) = 1$, so $f(v + tu) > 0$ when $0 < t < \delta$ for some $\delta > 0$. Let $0 < \varepsilon < \min\{f(x + u) - f(x + v)\ ,\ \delta\}/3$. Then we have

$$f(v + \varepsilon u) > 0, \tag{4}$$

$$f(x + u) > f(x + v) + 2\varepsilon. \tag{5}$$

Since $f(v + \varepsilon u) \leq f(v) + \varepsilon f(u) = 1 + \varepsilon$, that is, $f(v + \varepsilon u) - 1 \leq \varepsilon$. Since

$$1 = f(v) = f[(v + \varepsilon u) - \varepsilon u] \leq f(v + \varepsilon u) + \varepsilon f(u) = f(v + \varepsilon u) + \varepsilon.$$

Hence we get

$$|f(v + \varepsilon u) - 1| \leq \varepsilon. \tag{6}$$

This implies that

$$\begin{aligned} f(x - u) &= f[x - (\varepsilon + f(v + \varepsilon u))u + (\varepsilon + f(v + \varepsilon u) - 1)u] \\ &= f[x - (\varepsilon + f(v + \varepsilon u))u] + |\varepsilon + f(v + \varepsilon v) - 1| \\ &\leq f[x - (\varepsilon + f(v + \varepsilon u))u] + \varepsilon + |f(v + \varepsilon v) - 1| \\ &\leq f[x - (\varepsilon + f(v + \varepsilon u))u] + 2\varepsilon, \end{aligned}$$

that is,

$$f(x - u) - 2\varepsilon \leq f[x - (\varepsilon + f(v + \varepsilon u))u]. \tag{7}$$

By (4), let $x_0 = (x+\varepsilon u)/f(v+\varepsilon u)$ and $y = \varepsilon u/f(v+\varepsilon u)$. Since $0 \in P_f(x)$, we have $0 \in_f (x/f(v+\varepsilon u))$. So $y \in P_f(x_0)$. Obviously, $f(y) > 0$ and $y/f(y) = u$. By (7), we have

$$\begin{aligned} f(x_0 - y/f(y)) &= f(x_0 - u) \\ &= f[(x-\varepsilon u)/f(v+\varepsilon u) \ - \ u] \\ &= (1/f(v+\varepsilon u))f(x+\varepsilon u - f(v+\varepsilon u)u) \\ &\geq (f(x-u) - 2\varepsilon)/f(v+\varepsilon u) \\ &> f(x-v)/f(v+\varepsilon u) \\ &= f[(x+\varepsilon u) - (v+\varepsilon u)]/f(v+\varepsilon u) \\ &\geq f[(x+\varepsilon u)/f(v+\varepsilon u) - (v+\varepsilon u)/f(v+\varepsilon u)] \\ &= f[x_0 - (v+\varepsilon u)/f(v+\varepsilon u)]. \end{aligned}$$

Since $f[(v+\varepsilon u)/f(v+\varepsilon u)] = 1$. This is in contradiction with the assumption. Thus Y is an f-homogeneous imbedded subspace of X. □

Theorem 3.3 *Let X be a locally convex space and f a real function defined X which satisfies the condition (F1) and $f(0) = 0$. Then Y is an f-homogeneous imbedded subspace of X if and only if, for every $r > 0$, Y is a P_r-homogeneous imbedded subspace of X.*

Proof By the Theorem 2.3 of [5], Y is f-proximinal if and only if Y is P_r-proximinal for every $r > 0$.

Assume that Y is an f-homogeneous imbedded subspace of X. Given $r > 0$, let $x \in X$ and $u, v \in Y$ such that $0 \in P_{P_r}(x)$ and $P_r(u) = P_r(v)$. If $P_r(u) = 0$, evidently, $f(u) = f(v) = 0$. So we have $f(x+u) = f(x+v)$. Assume that $P_r(u) = \lambda \neq 0$. By the Lemma 2.1,

$$P_r(u) = \lambda P_{\psi(\lambda)r}(u) = \lambda P_{\psi(\lambda)r}(v). \tag{8}$$

By the difinition of λ, we have $P_{\psi(\lambda)r}(u) = P_{\psi(\lambda)r}(v) = 1$. By the Lemme 2.2 of [5], $f(u) = f(v) = \psi(\lambda)r$. By the Theorem 2.3 of [5], $0 \in P_f(x)$. Since Y is a f-homogeneous imbedded subspace, we have $f(x+u) = f(x+v)$. If $f(x+u) = 0$, obviously, $P_r(x+u) = P_r(x+v) = 0$. Suppose that $\alpha = f(x+u) > 0$. By the Lemma 2.2 of [5], we have $P_\alpha(x+u) = P_\alpha(x+v) = 1$. Let $\beta = \psi^{-1}(r/\alpha)$. Then $\beta > 0$ and $\psi(\beta)\alpha = r$. By Lemma 2.1,

$$\begin{aligned} P_r(x+u) &= P_{\psi(\beta)\alpha}(x+u) \\ &= \beta^{-1}P_\alpha(x-+u) \\ &= \beta^{-1}P_\alpha(x+v) \\ &= P_{\psi(\beta)\alpha}(x+v) \\ &= P_r(x+v). \end{aligned}$$

Thus Y is a P_r-homogeneous imbedded subspace.

Assume that, for every $r > 0$, Y is a P_r-homogeneous imbedded subspace. Let $x \in X$ and $u, v \in Y$ such that $0 \in P_f(x)$ and $f(u) = f(v)$. Obviously, if $f(u) = 0$, then $P_r(u) = 0$ for every $r > 0$. So $P_r(x+u) = Pr(x+v)$ for every $r > 0$. Assume that $r = f(u) > 0$. By the Lemma 2.2 of [5], $P_r(u) = P_r(v) = 1$. By Theorem 2.3 of [5], $0 \in P_{P_r}(x)$. Since Y is a P_r-homogeneous imbedded subspace, we have $P_r(x+u) = P_r(x+v)$. We may assume that $\lambda = P_r(x+u) > 0$. By Lemma 2.1, we have

$$P_r(x+u) = \lambda P_{\psi(\lambda)r}(x+u) = P_r(x+v) = \lambda P_{\psi(\lambda)r}(x+v).$$

So $P_{\psi(\lambda)r}(x+u) = \lambda P_{\psi(\lambda)r}(x+v) = 1$. By Lemma 2.1, one has $f(x+u) = \psi(\lambda)r = f(x+v)$. So Y is an f-homogeneous imbedded subspace of X. □

Remark By Theorem 3.3, if the condition (F2) is replaced by (F1) in Lemma 2.3, 2.4 and Theorem 3.1 and 3.2, we have the same conclusions respectively.

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局部凸空间中的齐次嵌入子空间

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在本文中，将齐次嵌入子空间概念引入了局部凸空间中，并讨论了它们了逼近性质.