行波的稳定性

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摘　要

文[1]引入一种扩充系统用来计算带 $O(2)$ 对称性的非线性常微分方程的行波(或回转波).本文证明,这些行波的稳定性由扩充系统的特征值来决定,其机理与通常的静态解的稳定性的判别相同.

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Stability of Travelling Waves *

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Abstract An extended system was introduced in [1] for the computation of travelling wave (or rotating wave) solutions of nonlinear ODE's with $O(2)$ symmetry. In this paper we show that the stability of the travelling waves can be determined by the eigenvalues of the extended system in a way similar to the situation for the stability of usual steady-state solutions.

Key words stability, travelling waves, nonlinear ODE, $O(2)$ symmetry, extended system

1. Introduction

This paper is concerned with the stability of travelling (or rotating) wave solutions of the ODE

$$\frac{du}{dt} = f(u,\lambda), \quad t \geq 0, \tag{1.1}$$

where f is a C^2 mapping from $U \times R$ into U and U a finite dimensional Hilbert space with inner product $\langle\cdot,\cdot\rangle$ (An extension to infinite dimensions can be done as in Section 5 of [8] or Section 6 of [1]). We assume that f is symmetric (equivariant, commutable) with respect to an action of the group $O(2)$ on U, that is,

$$\gamma f(u,\lambda) = f(\gamma u,\lambda), \forall\gamma \in O(2), \quad u \in U, \tag{1.2}$$

where $O(2)$ is generated by rotations $r_\alpha, \alpha \in \mathcal{R}$, and a reflection s satisfying for all $\alpha, \beta \in \mathcal{R}$ (where 1 stands for the group identity):

$$r_{\alpha+2\pi} = r_\alpha, \quad r_{\alpha+\beta} = r_\alpha r_\beta = r_\beta r_\alpha, \tag{1.3a}$$

$$s^2 = r_0 = r_{2\pi} = 1, \quad s r_\alpha = r_{-\alpha} s, \tag{1.3b}$$

$$\langle\gamma u, \gamma v\rangle = \langle u, v\rangle, \quad \forall\gamma \in O(2), \quad u, v \in U. \tag{1.3c}$$

A travelling wave solution (TW for short) of (1.1) is a special periodic solution in the form

$$v(t) = r_{\omega t} u, \tag{1.4}$$

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where $w \in \mathcal{R}$, the velocity of the wave, and $u \in U$ are independent of time t.

TW is one of the two most common periodic solutions of (1.1) (the other is the standing wave solutions, cf. [2]). Its bifurcation is studied by many authors ([1],[2],[3],[8]). In particular, results on its stability can be found in e.g. [4] and [5]. In this paper we re-attackle the stability problem through a new approach in terms of an extended system (see (2.13) below) introduced in [1]. This extended system (ES in short) takes into account the existence of the solution orbits due to the $O(2)$ symmetry and the fact that TW satisfies a steady-state equation (see (2.4) below, also [1]). The virtue of ES for the computation of TW has been demonstrated in [1]. We show in this paper that ES is also good for stability analysis in that the (orbital) stability of TW can be determined simply by the eigenvalues of ES very similar to the situation for the stability of the usual steady-state solutions of nonlinear ODE's. Our results will prove useful when one considers the stability in bifurcation problems of TW. See e.g. [7], where a kind of period-doubling bifurcation of TW is discussed.

In the next section we provide some preliminary material. In Section 3 we give the main results. We shall show that ES always has eigenvalues $\pm\sqrt{C_0}$ for certain constant $C_0 \in R$. TW is stable if all the other eigenvalues of ES have negative real parts, and unstable if an eigenvalue other than $\pm\sqrt{C_0}$ has a positive real part. The crucial point is an explicit expression (see (3.3)) of the Floquet operator of TW which determines the stability.

2. Preliminaries

Let us define

$$r'_\alpha x := \frac{\partial}{\partial \alpha}(r_\alpha x), \tag{2.1}$$

$$A := r'_0. \tag{2.2}$$

The linear operator A will play an important role in our analysis. By (1.3) it is easy to deduce that (cf. Lemma 2.1, [1]):

$$r'_\alpha = r_\alpha A, \quad r_\alpha A = A r_\alpha, \quad sA = -As. \tag{2.3}$$

A direct consequence of (2.3) and the equivariance condition (1.2) is that (cf. (2.5) in [1] and Lemma 1.1, [5]) $(v(t), \lambda) = (r_{wt}u, \lambda)$ is TW if and only if (u, w, λ) satisfies

$$\tilde{f}(u, w, \lambda) := f(u, \lambda) - wAu = 0. \tag{2.4}$$

Hence we also call (u, w, λ) a TW if it solves (2.4). The significance of this observation is that (2.4) is a "steady-state" equation and so it is amenable to the standard steady-state bifurcation theory. We notice that $\tilde{f}$ defined by (2.4) is equivariant with respect to r_α but not s for $w \neq 0$. This accords with the fact that TW breaks the reflection symmetry while preserves the rotation symmetry.

Note that if (u, w, λ) is TW then so are $(r_\alpha u, w, \lambda)$ for all $a \in \mathcal{R}$ due to (2.3),(2.4) and (1.2), which form a solution orbit generated by (u, w, λ). Now inserting $(r_\alpha u, w, \lambda)$ into (2.4) and differentiating it in α at $\alpha = 0$ yields

$$\tilde{f}_u(u, w, \lambda)Au = 0. \tag{2.5}$$

Hence we always have

$$Au \in \mathrm{Null}(\tilde{f}_u(u, w, \lambda)) \tag{2.6}$$

so long as (u, w, λ) is TW. In face Au is the tangent vector of the solution orbit $(r_\alpha u, w, \lambda)$ at $\alpha = 0$. We always assume

$$Au \neq 0. \tag{2.7}$$

We remark that

$$Au \neq 0 \Leftrightarrow u \notin \mathrm{Fix}(SO(2)), \tag{2.8}$$

where $SO(2) \subset O(2)$ is the subgroup generated by rotations $r_\alpha, a \in \mathcal{R}$, and $\mathrm{Fix}(\Sigma)$ denotes the fixed point subspace of U for any subgroup Σ of $O(2)$:

$$\mathrm{Fix}(\Sigma) := \{u \in U, \quad \gamma u = u, \forall \gamma \in \Sigma\}. \tag{2.9}$$

(2.8) can be easily deduced by noting (2.3). We also always assume

$$w \neq 0. \tag{2.10}$$

namely, we are interested in TW with nonzero velocity.

In order to isolate a point on the orbit we require

$$l_0 u = 0, \tag{2.11}$$

where l_0 is some linear functional on U satisfying

$$l_0 Au \neq 0. \tag{2.12}$$

Putting (2.11) and (2.4) together leads to the extened system (see [1])

$$F(x, \lambda) := \begin{pmatrix} f(u, \lambda) - wAu \\ l_0 u \end{pmatrix} = 0, \tag{2.13}$$

$$x = (u, w), F : X \times R \to X := U \times R.$$

This is the extended system which will play a central role in our analysis.

We end this section with a simple lemma. Its proof is simple and omitted.

Lemma 2.1 *If* (u, w, λ) *is* TW *and*

$$Null(\tilde{f}_u(u, w, \lambda)) = span(Au), \tag{2.14}$$

$$Au \notin Range(\tilde{f}_u(u, w, \lambda)), \tag{2.15}$$

then the Jacobian $F_x(x, \lambda)$ *is an isomorphism and hence there exists a unique solution path* $(\tilde{x}(\tilde{\lambda}), \tilde{\lambda})$ *of (2.13) with* $\tilde{x}(\lambda) = x = (u, w)$*, corresponding to a unique (up to the solution orbit) path of* TW *near* (u, w, λ).

3. Stability of TW

In this section we first point out in Lemma 3.1 that the Floquet operator (or the monodromy operator) M of TW can be expressed explicitly in terms of $\tilde{f}_u := f_u - wA$. Then Lemma 3.2 reveals the relationship between the eigenvalues of $\tilde{f}_u$ and F_x. Finally we present the main Theorem 3.3 concerning the stability of TW as a consequence of Lemmas 3.1 and 3.2.

Let $v(t) = r_{wt}u$ be TW with a (minimum) period T. The corresponding Floquet operator can be defined as (see Section 7.2 [6]):

$$M := \Phi(T), \tag{3.1}$$

where $\Phi(t) : R \to L(U)$ solves the matrix initial value problem

$$\frac{d\Phi}{dt} = f_u(v(t), \lambda)\Phi, \quad \Phi(0) = I. \tag{3.2}$$

Remark On the period T. $v(t) = r_{wt}u$ obviously has period $2\pi/w$. But it may has smaller period. For instance let Z_m be the subgroup of $O(2)$ generated by $r_{2\pi/m}$. If $u \in \mathrm{Fix}(Z_m)$ for some $m \in Z^+$ then $T = 2\pi/(mw)$. This is the case when e.g. (u, w, λ) is on a TW path bifurcating from a trivial steady-state solution $(u = 0)$ path, or from a nontrivial steady-state solution path belonging to $\mathrm{Fix}(D_m)(D_m$ is the subgroup generated by Z_m and $s)$; See e.g. [8] and [1] respectively.

Lemma 3.1 *Let T be the period of TW $v(t) := r_{wt}u$. Then the corresponding Floquet operator*

$$M = r_{wT}e^{T\tilde{f}_u} = e^{T\tilde{f}_u}r_{wT}, \tag{3.3}$$

where $\tilde{f}_u = f_u(u, \lambda) - wA$.

Proof Set $\tilde{\Phi} := r_{-wt}\Phi$. Then by (3.2)

$$\frac{d\tilde{\Phi}}{dt} = \tilde{f}_u\tilde{\Phi}, \quad \tilde{\Phi}(0) = I. \tag{3.4}$$

But $\tilde{f}_u$ is independent of time t, so we may solve (3.4) to obtain

$$\tilde{\Phi}(t) = e^{t\tilde{f}_u}. \tag{3.5}$$

Thus

$$M = \Phi(T) = r_{wT}\tilde{\Phi}(T) = r_{wT}e^{T\tilde{f}_u}. \tag{3.6}$$

Finally we note

$$r_{wT}u = v(T) = v(0) = u. \tag{3.7}$$

So r_{wT} is comutable with $\tilde{f}_u$ and hence with $e^{T\tilde{f}_u}$. This shows the last equality of (3.3) and completes the proof.

In the sequel we denote by $\sigma(B)$ the totality of eigenvalues of a linear operator B.

Lemma 3.2 *Assume that (2.14) and (2.15) hold, q is the dimension of U and*

$$\sigma(\tilde{f}_u) = \{\alpha_1, \alpha_2, \cdots, \alpha_q\}. \tag{3.8}$$

Then,

(i) there is always one eigenvalue, say, $\alpha_1 = 0$ with the null vector Au;

(ii) $\alpha_j \neq 0$ for $j > 1$ and

$$\sigma(F_x) = \{\sqrt{C_0}, -\sqrt{C_0}, \alpha_2, \cdots, \alpha_q\}, \tag{3.9}$$

where $F_x := F_x(x, \lambda), x = (u, w)$ and

$$C_0 := l_0 Au. \tag{3.10}$$

Proof For (i) see (2.5). To show (ii) we first note that (2.14) and (2.15) imply $\alpha_j \neq 0$ for $j > 1$. Next we prove (3.9) in a constructive manner. Let α be a non-zero eigenvalue of $\tilde{f}_u$ with an eigenvector ϕ. If $\alpha^2 \neq C_0$ then we set

$$\theta := (\theta_1, \theta_2) \in X, \tag{3.11}$$

$$\theta_1 := \phi + \alpha^{-1}\theta_2 Au, \tag{3.12a}$$

$$\theta_2 := l_0\phi/(\alpha - \alpha^{-1}C_0). \tag{3.12b}$$

This makes

$$F_x\theta = \alpha\theta. \tag{3.13}$$

For the case $\alpha^2 = C_0$ the definition (3.12b) fails. Instead we define

$$\theta_1 := \alpha_{-1}Au, \quad \theta_2 := 1 \tag{3.14}$$

to validate (3.13). Therefore we have proved

$$\{\alpha_2, \cdots, \alpha_q\} \subset \sigma(F_x). \tag{3.15}$$

As in (3.14) set

$$\theta_1 := Au, \quad \theta_2 := \pm\sqrt{C_0}, \tag{3.16}$$

then

$$F_x\theta = \pm\sqrt{C_0}\theta. \tag{3.17}$$

(3.15) and (3.17) imply

$$\{\pm\sqrt{C_0}, \alpha_2, \cdots, \alpha_1\} \subset \sigma(F_x). \tag{3.18}$$

For the other direction of the proof we start with any $\alpha \in \sigma(F_x)$ and its eigenvector $\theta = (\theta_1, \theta_2) \neq 0$:

$$\tilde{f}_u\theta_1 + Au\theta_2 = \alpha\theta_1, \tag{3.19a}$$

$$l_0\theta_1 = \alpha\theta_2. \tag{3.19b}$$

Let us first consider the case $\tilde{f}_u\theta_1 = 0$. In this case $\theta_1 \neq 0$, since if $\theta_1 = 0$ then $\theta_2 \neq 0$ too by (3.19a). Hence $\theta_1 = CAu$ for some nonzero constant C. Then (3.19) gives $\alpha_2 = C_0$ or

$$\alpha \in \{\pm\sqrt{C_0}\}. \tag{3.20}$$

For the other case $\phi := \tilde{f}_u\theta_1 \neq 0$ we have $\tilde{f}_u\phi = \alpha\phi$ by (3.19a). Also observe that $\alpha \neq 0$ by (3.19),(2.14) and (2.15). Then we end up with

$$\alpha \in \{\alpha_2, \cdots, \alpha_q\}. \tag{3.21}$$

(3.20) and (3.21) lead to

$$\sigma(F_x) \subset \{\pm\sqrt{C_0}, \alpha_2, \cdots, \alpha_q\}. \tag{3.22}$$

(3.9) finally follows from (3.18) and (3.22).

Now we are ready to consider the stability of TW $v(t) := r_{wt}u$. Taking the existence of the solution orbit into account we adopt the definition of *orbital asymptotic stability* as in [2] and [5]. According to this definiton $v(t)$ is stable if any solution $\tilde{u}(t)$ of (1.1) which is close to $v(t)$ at $t = 0$ will approach the solution orbit $\{r_\alpha v(t), \alpha \in \mathcal{R}\}$ as $t \to \infty . v(t)$ is unstable if it is not stable. It is well-knowm (see [2] or [6]) that the stability of $v(t)$ may be determined by the eigenvalues of the corresponding Floquet operator M and hence, in virtue of Lemma 3.1 and 3.2, by $\sigma(\tilde{f}_u)$ or $\sigma(F_x)$. More precisely, we have the following main theorem.

Theorem 3.3 *Assumptions and notations as in Lemmas 3.1 and 3.2. Then,*

(i) $MAu = Au$.

(ii) $M : U_1 \to U_1 := Range(\tilde{f}_u)$.

(iii) TW $v(t)$ is stable if one of the following three equivalent statements holds:

(a) all eigenvalues of the restriction $M_1 := M|U_1$ have modulus less than 1;

(b) eigenvalues $\alpha_2, \cdots, \alpha_q$ of $\tilde{f}_u$ have negative real parts;

(c) all eigenvalues except $\pm\sqrt{C_0}$ of F_x have negative real parts.

(iv) $v(t)$ is untable if the following equivalent statements hold:

(a) an eigenvalue of M has modulus greater than 1;

(b) an eigenvalue of $\tilde{f}_u$ has positive real part;

(c) an eigenvalue of F_x other than $\pm\sqrt{C_0}$ has positive real part.

Proof (i) is a direct consequence of (3.3) and (2.14). To show (ii) let $y \in U_1$ then there exists a $\tau \in U$ such that

$$y = \tilde{f}_u\tau.$$

But M is commutable with $\tilde{f}_u$ since both r_{wT} and $e^{T\tilde{f}_u}$ are (cf. (3.3)). So

$$My = \tilde{f}_u M\tau.$$

(ii) then follows.

By (i) and (ii) we have

$$\sigma(M) = \{1, \sigma(M_1)\}. \tag{3.23}$$

Then a simple application of Theorem 6.2, Ch. XVI [2] shows that (iii,a) leads to the stability of $v(t)$.

The equivalence of (iii,b) and (iii,c) has been established in Lemma 3.2. Next we try to confirm the equivalence of (iii,a) and (iii,b). We observe that $M, e^{T\tilde{f}_u}$ and r_{wT} are all commutable with each other and they map U_1 into itself. So we may define the restriction operator

$$E := e^{T\tilde{f}_u}|U_1, \tag{3.24}$$

and this gives

$$M_1 = r_{wT}E. \tag{3.25}$$

Now let $\|\cdot\|$ denote the induced matrix norm from the inner product $\langle\cdot,\cdot\rangle$ within the subspace U_1. Then by (1.3a) and (1.3c)

$$\|r_\alpha^n\| = \|r_{n\alpha}\| = 1, \forall\alpha \in R, n \in Z. \tag{3.26}$$

It follows from (3.25) and (3.26) that

$$\|M_1^n\| = \|r_{wT}^n E^n\| \leq \|r_{nwT}\|\|E^n\| = \|E^n\| = \|r_{-nwT}M_1^1\| \leq \|M_1^n\|.$$

So

$$\|M_1^n\| = \|E_1^n\|. \tag{3.27}$$

The desired equivalence then follows immediately:

$$(\text{iii}, \alpha) \Leftrightarrow \|M^n\| \to 0 \Leftrightarrow \|E^n\| \to 0 \Leftrightarrow (\text{iii}, b). \tag{3.28}$$

This completes the proof to (iii). The proof to (iv) is similar and ommitted.

Remark 3.2 Recall that we may use the extended system $F(x, \lambda) = 0$ to follow a path of TW. Theorem 3.3 enables us to monitor the change of stability of TW by detecting an imaginary axis crossing of an eigenvalue of F_x just as we shall do for "ordinary" steady-state bifurcations (cf. [7]).

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