base. Then $(Y, \mathcal{U})$ is not locally symmetric. Define

$$F_n(x) = \begin{cases} \{x\}, & x \le n \\ \{1\}, & x > n \end{cases} \quad \text{for all } x \in X \text{ and } F(x) = \{x\} \text{ for all } x \in X.$$

Then $\{F_n : n \in N\}$ is a net of continuous multifunctions and converges quasiuniformly to F, but F is not continuous. This also serves as a counterexample to Theorem 3.5 and Theorem 3.6 of Naimpally [3].

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集值映射的拟一致收敛

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摘 要

本文中，在拟一致空间中引入强局部对称性的概念，并讨论集值映射的拟一致收敛性．本文推广了[5] 中的一些结果，同时用反例否定了[3] 和[4] 中的主要定理．

Journal of Mathematical Research & Exposition
Vol.14, No.3, Aug. 1994

Quasiuniform Convergence for Multifunctions *

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Abstract We introduce the strongly local symmetry in quasiuniform spaces and discuss quasiuniform convergence for multifunctions. Some results of [5] are generalized and counterexamples are given to show that the main theorems of [3] and [4] are false.

Key words oquasiuniform space, strongly local symmetly, quasiuniform convergence.

It is well known that the pointwise limit f of a sequence $\{f_n : n \in N\}$ of continuous functions, need not be continuous. In 1841, Weierstrass discovered uniform convergence which provides a sufficient condition for f to be continuous. Later on, some of these ideas were generalized in the setting of uniform spaces in [2] . In 1971, Smithson [5] generalized some results of [2] to multifunctions. In this short paper, we will generalize some results of [5] to quasiuniform spaces by introducing the concept of strongly local symmetry.

Let $(Y, \mathcal{U})$ be a quasiuniform space (all axioms for a uniform space hold except for the symmetry axiom).

Dedinition 1[1] *$(Y, \mathcal{U})$ is said to be locally symmetric if for each $U \in \mathcal{U}$ and each $y \in Y$ there is a symmetric $V \in \mathcal{U}$ such that $V^2(y) \subset U(y)$.*

Definition 2 *$(Y, \mathcal{U})$ is said to be strongly locally symmetric if for each $U \in \mathcal{U}$ and each $y \in Y$ there are a symmetric $V \in \mathcal{U}$ and an open subset G containing y such that $V^2(z) \subset U(z)$ for each $z \in G$.*

It is obvious that a strongly local symmetric quasiuniform space is locally symmetric, but the converse is not true.

Definition 3[3] *A net in $(Y, \mathcal{U})$ is called a Cauchy net if for each $U \in \mathcal{U}$ there is a point $y_0 \in Y$ such that $\{y_\alpha : \alpha \in D\}$ is eventually in $U(y_0)$. And $(Y, \mathcal{U})$ is complete if each Cauchy net in $(Y, \mathcal{U})$ converges to some point in Y.*

Definition 4[4] *A sequence $\{y_n : n \in N\}$ in $(Y, \mathcal{U})$ is said to be $\mathcal{U}$-Cauchy if for each $U \in \mathcal{U}$ there is an $m \in N$ such that for all $n > m, y_n \in U(y_m)$. And $(Y, \mathcal{U})$ is said to be sequentially complete if every $\mathcal{U}$-Cauchy sequence converges to a point in Y.*

From the above definitions, it is obvious that a complete quasiuniform space is sequentially complete. Recall that a multifunction $F : X \to Y$ from a space X to a space Y is

*Received Mar. 2, 1992.

upper semicontinuous (u. s. c.), lower semicontinuous (l. s. c.) if for each $x \in X$ and each open W with $F(x) \subset W(F(x) \cap W \neq \emptyset)$ there is an open G containing x such that $F(G) \subset W(F(z) \cap W \neq \emptyset$ for each $z \in G$, and $F : X \to Y$ is continuous if and only if it is both u.s.c. and l.s.c.. Throughout the paper, $Y^{mX}(Y^X)$ denotes the family of all multifunctions (functions) from X to Y.

Definition 5[6] *A multifunction $F : X \to Y$ is called weakly semicontinuous (w.u.s.c.), (weakly lower semicontinuous (w.l.s.c.)) if for each $x \in X$ and each open W with $F(x) \subset W(F(x) \cap W \neq \emptyset)$ there is an open G containing x such that $F(G) \subset \bar{W}(F(z) \cap \bar{W} \neq \emptyset)$ for each $z \in G$, where $\bar{W}$ is the closure of W in Y. And F is weakly continuous if and only if it is both w.u.s.c. and w.l.s.c.. Obviously, a continuous multifunction is weakly continuous.*

Now let X be a space, $(Y, \mathcal{U})$ be a quasiuniform space. For each $U \in \mathcal{U}$, set

$$W(U) = \{(F, H) : H(x) \subset U(F(x)) \text{ and } F(x) \subset U^{-1}(H(x)) \text{ for all } x \in X\}.$$

Let $\mathcal{W}$ denote the quasiuniformity on Y^{mX} generated by taking $\{W(U) : U \in \mathcal{U}\}$ as a base. The topology on Y^{mX} generated by $\mathcal{W}$ is called the topology of quasiuniform convergence. And a net of multifunctions $\{F_\alpha : \alpha \in D\}$ is said to converge quasiuniformly to F if and only if it converges to F in the topology of quasiuniform convergence.

Theorem 1 *Let $\{F_\alpha : \alpha \in D\}$ be a net of weakly continuous multifunctions of a space X to a locally symmetric quasiuniform space $(Y, \mathcal{U})$, and $F \in Y^{mX}$ be point compact. If $\{F_\alpha : \alpha \in D\}$ converges quasiuniformly to F, then F is continuous.*

Proof Let $x \in X$ and W be open with $F(x) \subset W$, choose a symmetric $U \in \mathcal{U}$ such that $U^4(F(x)) \subset W$. Since $\{F_\alpha : \alpha \in D\}$ converges quasiuniformly to F, there is a $\beta \in D$ such that $F_\alpha(z) \subset U(F(z))$ and $F(z) \subset U^{-1}(F_\alpha(z))$ for each $z \in X$ whenever $\alpha > \beta$. For a fixed $\alpha > \beta$, since F_α is w.u.s.c., there is an open G containing x such that $F_\alpha(G) \subset \overline{U(F_\alpha(x))}$. For each $z \in G$ and each $y \in F_\alpha(z)$, there must be a point $y' \in U(y) \cap U(F_\alpha(x))$, hence $y \in U(y') \subset U^2(F_\alpha(x))$ and

$$F(z) \subset U(F_\alpha(z)) \subset U^3(F_\alpha(x)) \subset U^4(F(x)) \subset W.$$

It follows that $F(G) \subset W$. Therefore F is u.s.c..

Now suppose $F(x) \cap W \neq \emptyset$, and let $y \in F(x) \cap W$. Choose a symmetric $U \in \mathcal{U}$ such that $U^3(y) \subset W$. Let $\beta \in D$ such that $F_\alpha(z) \subset U(F(z))$ and $F(z) \subset U^{-1}(F_\alpha(z))$ for each $z \in X$ and all $\alpha > \beta$. For a fixed $\alpha > \beta$, since $F_\alpha(x) \cap U(y) \neq \emptyset$ and F_α is w.l.s.c., there is an open G containing x such that $F_\alpha(z) \cap \overline{U(y)} \neq \emptyset$ for each $z \in G$. For each $z \in G$, choose $y' \in F_\alpha(z) \cap \overline{U(y)}$ and $y'' \in U(y') \cap U(y)$, then there is a $y_0 \in F(z)$ such that $y' \in U^{-1}(y_0)$. Therefore $y_0 \in U(y') \subset U^2(y'') \subset U^3(y) \subset W$. It follows that $F(z) \cap W \neq \emptyset$ for each $z \in$G. So F is l.s.c..

Corollary[5] Let $\{F_\alpha : \alpha \in D\}$ be a net of continuous multifuntions from a space X to a uniform space $(Y, \mathcal{U})$. If $\{F_\alpha : \alpha \in D\}$ converges uniformly to a point compact multifunction F, then F is continuous.

It is evident to see that the topology of pointwise convergence is coarser than the topology of quasiuniform convergence. The following theorem will establish the relationship between the compact open topology and the topology of quasiuniform convergence.

Theorem 2 *Let $\mathcal{F}$ be a family of point compact continuous multifunctions from a space X to a quasiuniform space $(Y, \mathcal{U})$.*

(i) If $(Y, \mathcal{U})$ is locally symmetric, then the compact open topology is coarser than the topology of quasiuniform convergence on $\mathcal{F}$.

(ii) If X is compact and $(Y, \mathcal{U})$ is strongly locally symmetric, then the compact open topology and the topology of quasiuniform convergence are the same on $\mathcal{F}$.

Proof (i) Let K be compact in X and G be open in $(Y, \mathcal{U})$, it is sufficient to show that both $T_1(K,G) = \{F \in \mathcal{F} : F(K) \subset G\}$ and $T_2(K,G) = \{F \in \mathcal{F} : F(x) \cap G \neq \emptyset$ for all $x \in K\}$ are open in the topology of quasiuniform convergence. For each $F \in T_1(K,G)$, there is a $U_1 \in \mathcal{U}$ such that $U_1(F(K)) \subset G$, then $W(U_1)(F) \subset T_1(K,G)$. Hence $T_1(K,G)$ is open in the topology of quasiuniform convergence. On the other hand, since $(Y, \mathcal{U})$ is regular, by Lemma 4 of [5], for each $F \in T_2(K,G)$, there is a compact subset C of $(Y, \mathcal{U})$ such that $C \subset F(K) \cap G$ and $F(x) \cap C \neq \emptyset$ for each $x \in K$. Choose a $U_2 \in \mathcal{U}$ such that $U_2(C) \subset G$. Therefore $W(U_2)(F) \subset T_2(K,G)$ and $T_2(K,G)$ is open in the topology of quasiuniform convergence.

(ii) It is sufficient to show that the topology of quasiuniform covergence is coarser than the compact open topology. To this end, for each $U \in \mathcal{U}$ and each $F \in \mathcal{F}$. Since $(Y, \mathcal{U})$ is strongly locally symmetric, for each $x \in X$, there is a symmetric $V_x \in \mathcal{U}$ such that $V_x^2(y) \subset U(y)$ for each $y \in F(x)$. Since $F(x)$ is compact, there are finitely many points $y_i \in F(x)$ $(1 \leq i \leq n(x))$ such that

$$F(x) \subset \bigcup_{i=1}^{n(x)} \mathrm{Int}(V_x(y_i)) \text{ and } F(x) \cap \mathrm{Int}(V_x(y_i)) \neq \emptyset, \text{ for each } 1 \leq i \leq n(x).$$

Furthermore, by the continuity of F, we can choose a compact neighborhood K_x of x such that

$$F(K_x) \subset \bigcup_{i=1}^{n(x)} \mathrm{Int}(V_x(y_i)) \text{ and } F(z) \cap \mathrm{Int}(V_x(y_i)) \neq \emptyset,$$

where $z \in K_x$, $1 \leq i \leq n(x)$.

Since X is compact, there are finitely many points $x_1, x_2, \cdots, x_m$ such that $X = \bigcup_{j=1}^{m} K_{x_j}$. Now set

$$G_j = \bigcup_{i=1}^{n(x)} \mathrm{Int}(V_{x_j}(y_i)), \text{ for each } 1 \leq j \leq m,$$

and

$$T(K_{x_j}, G_j) = \{H \in \mathcal{F} : H(K_{x_j}) \subset G_j, \quad H(x) \cap \mathrm{Int}(V_{x_j}(y_i)) \neq \emptyset,$$

where $x \in K_{x_j}$, $1 \leq i \leq n(x_j)$.

Then $F \in \bigcap_{j=1}^{m} T(K_{x_j}, G_j) \subset W(U)(F)$. Therefore the topology of quasiuniform convergence is coarser than the compact open topology.

Corollary[5] Let $\mathcal{F}$ be a family of point compact continuous multifunctions from a compact Hausdorff space to a uniform space $(Y, \mathcal{U})$. Then the topology of uniform convergence is the same as the compact open topology on $\mathcal{F}$.

Theorem 1 provides a sufficient condition under which the limit of a net of continuous multifunctions is continuous. Now we will discuss some necessary and sufficient condition. Recall a net $\{F_\alpha : \alpha \in D\}$ of multifunctions converges pointwisely to F if it converges to F in the topology of pointwise convergence.

Definition 6 *A net of multifunctions $\{F_\alpha : \alpha \in D\}$ from a space X to a quasiuniform $(Y, \mathcal{U})$ converges pseudo-quasiuniformly to F if it converges pointwisely to F and for each $U \in \mathcal{U}$ and each $\alpha \in D$, there are finitely many $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that for each $x \in X$ there are α_i and α_j $(1 \le i, j \le n)$ satisfying $F_{\alpha_i}(x) \subset U(F(x))$ and $F(x) \subset U^{-1}(F_{\alpha_j}(x))$.*

The proof of the following theorem is very similar to the proof of Theorem 1, so we omit it.

Theorem 3 *Let $\{F_\alpha : \alpha \in D\}$ be a net of continuous multifunctions from a compact Hausdorff space X to a strongly locally symmetric quasiuniform space $(Y, \mathcal{U})$, and $F \in Y^{mX}$ be point compact. Then $\{F_\alpha : \alpha \in D\}$ converges pseudo-quasiuniformly to F if and only if F is continuous and $\{F_\alpha : \alpha \in D\}$ converges pointwisely to F.*

Definition 7[4] *A sequence of functions $\{f_n : n \in N\}$ from a space X to a quasiuniform space $(Y, \mathcal{U})$ is called $\mathcal{U}$-Cauchy if for each $U \in \mathcal{U}$ there is a $n_0 \in N$ such that for all $x \in X$ and each $n > n_0, (f_{n_0}(x), f_n(x)) \in U$.*

Theorem 6 of [4] say that every $\mathcal{U}$-Cauchy sequence of functions from X to a Hausdorff sequentially complete space $(Y, \mathcal{U})$ converges quasiuniformly. Similarly, Theorem 2.10 of [3] say that if $(Y, \mathcal{U})$ is a Hausdorff complete quasiuniform space, then $(Y^X, \mathcal{W})$ is complete. Now we give a counterexample to these theorems.

Example 1 Let $X = Y = N$, $U_n = \{(y, z) : y = z$ or $y = 1, z \ge n\}$. Let $\mathcal{U}$ be the quasiuniformity generated by taking $\{U_n : n \in N\}$ as a base on Y. Then $(Y, \mathcal{U})$ is Hausdorff complete. Define

$$f_n(x) = \begin{cases} x, & x \le n \\ 1, & x > n \end{cases}$$

for each $n \in N$ and each $x \in X$, then $\{f_n : n \in N\}$ is a Cauchy net, but it has no limit in $(Y^X, \mathcal{W})$.

The following example will show that Theorem 1 doesn't hold if $(Y, \mathcal{U})$ is not locally symmetric.

Example 2 Let $X = Y = N, B_n(1) = \{1\} \cup \{x \in X : x > n\}$ for each $n \in N, B_n(i) = \{i\}$ for all $i > 1$ and for each $n \in N$. X can be topologized by the neighborhood system $\{\mathcal{B}(x) : x \in X\}$, where $\mathcal{B}(x) = \{B_n(x) : n \in N\}$ for each $x \in X$. Let $U_n = \{(y, z) : y = z$ or $y > n, z \in Y\}$, $\mathcal{U}$ be the quasiuniformity on Y generated by taking $\{U_n : n \in N\}$ as a base. Then $(Y, \mathcal{U})$ is not locally symmetric. Define

$$F_n(x) = \begin{cases} \{x\}, & x \le n \\ \{1\}, & x > n \end{cases} \quad \text{for all } x \in X \text{ and } F(x) = \{x\} \text{ for all } x \in X.$$

Then $\{F_n : n \in N\}$ is a net of continuous multifunctions and converges quasiuniformly to F, but F is not continuous. This also serves as a counterexample to Theorem 3.5 and Theorem 3.6 of Naimpally [3].

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摘 要

本文中, 在拟一致空间中引入强局部对称性的概念, 并讨论集值映射的拟一致收敛性. 本文推广了[5] 中的一些结果, 同时用反例否定了[3] 和[4] 中的主要定理.