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The Compactness of Block Diagonal Operators *

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Let $\mathcal{H}$ be a comples separable Hilbert space and $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. Let $\mathcal{K}_+ = \sum_{i=0}^{+\infty} \oplus H$ and $\mathcal{K} = \sum_{n=-\infty}^{+\infty} \oplus H$. In this note, we consider a block diagonal operator $D = \sum_{i=0}^{+\infty} \oplus A_n$ (respectively $D = \sum_{n=-\infty}^{+\infty} \oplus A_n$) on $\mathcal{K}_+$ (respectively $\mathcal{K}$, where $A_n \in \mathcal{B}(\mathcal{H})$ for each n and $\sup\|A_n\| < +\infty$.

When $\dim\mathcal{H} = 1$, it is known that D is a compact operator if and only if $\lim\limits_{n\to\infty} a_n = 0$, where A_n is replaced by a scalar a_n (see [1], Problem 171). In general terms, we have come to following conclusion.

Theorem *A block diagonal operator $D = \sum_{i=0}^{+\infty} \oplus A_n$ or $D = \sum_{n=-\infty}^{+\infty} \oplus A_n$ is compact if and only if its every diagonal element A_n is compact and $\lim\limits_{n\to\infty} \|A_n\| = 0$.*

By the above theorem, we can obtain several corollaries as follows.

Corollary 1 *If $\mathcal{H}$ is an infinite dimensional Hilbert space and there exists a diagonal element A_n which is invertible, then the block diagonal operator D can't be compact.*

An operator $S \in \mathcal{B}(\mathcal{K}_+)$ is called a unilateral operator weighted shift with the weight sequence $\{A_n\}_0^{+\infty}$ if

$$S(x_0, x_1, \cdots) = (0, A_0x_0, A_1x_1, \cdots), \quad \forall (x_n) \in \mathcal{K}_+,$$

which is denoted by $S \sim \{A_n\}_0^{+\infty}$. Analogously, we can define a bilateral operator weighted shift $S \sim \{A_n\}_{-\infty}^{+\infty}$ on $\mathcal{K}$.

Corollary 2 *Let $S \sim \{A_n\}$ is an operator weighted shift (unilateral or bilateral), then S is compact if and only if every weight A_n is compact and $\lim\limits_{n\to\infty} \|A_n\| = 0$.*

Corollary 3 *A compact unilateral or bilateral operator weighted shift $S \sim \{A_n\}$ belongs to Schatten p-class if and only if $\sum\sum \lambda_{nm}^p < +\infty$, where $\lambda_{n1}, \lambda_{n2}, \cdots, \lambda_{nm}, \cdots$ are the eigenvalues of $[A_n] = (A_n^*A_n)^{1/2}$, each repeated as often as its multiplicity.*

References

[1] P.R. Halmos, *A Hilbert Space Problem Book*, 2nd. ed. Springer-Verlag, 1982.

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