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集值映象的变分不等式

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摘　要

本文应用Ky Fan 定理和KKM 技巧在(局部凸)Hausdorff 拓扑向量空间及自反Banach空间上讨论了集值映象的变分不等式解的存在性。所讨论的问题比[6,7,10,11] 中讨论的更为广泛。

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Variational Inequalities for Set-Valued Maps *

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Abstract This paper deals with the existence of solution to variational inequalities for set-valued maps by Ky Fan theorems and technique of KKM in a (locally convex) Hausdorff topological vector space and a reflexive Banach space. The problems discussed here are more general than that in [6, 7, 10, 11].

Key words variational inequality, KKM.

1. Introduction

Let E be a Hausdorff topological vector space (t.v.s.) or locally convex Hausdorff topological vector space (l.c.s.) with dual E', and $X \subset E$ a nonempty closed convex set. We denote the pairing between E' and E by $\langle w, u\rangle$ for $w \in E', u \in E$.

I. Let $X \subset E$ be nonempty compact and $a : E \times E \to R$ be a continuous bilinear form, i.e., there exists a constant $\beta \geq 0$ such that

$$a(u,v) \leq \beta\|u\|\|v\| \quad \text{for all} \quad u, v \in E.$$

Let the form $b : E \times E \to R$ satisfy the following conditions:
(i) $b(u,v)$ is continuous in u; or (i$'$) $b(u,v)$ is continuous linear in u;
(ii) $b(u,v)$ is convex lower semicontinuous in v.
Suppose that $T : E \to 2^{E'}$ is a set-valued map. We shall deal with the following:
Problem 1 Find $\bar{u} \in X, \bar{w} \in T\bar{u}$ such that

$$a(\bar{u}, v-\bar{u}) + b(\bar{u}, v) - b(\bar{u}, \bar{u}) \geq \langle \bar{w}, v-\bar{u}\rangle \quad \text{for all} \quad v \in X. \tag{1}$$

Problem 2 Find $\bar{u} \in X$ such that

$$a(\bar{u}, v-\bar{u}) + b(\bar{u}, v) - b(\bar{u}, \bar{u}) \geq \sup_{w\in T\bar{u}} \langle w, v-\bar{u}\rangle \quad \text{for} \quad v \in X. \tag{2}$$

II. When E is a reflexive Banach space and $X \subset E$ is a closed convex set, let $a(u,v)$ be coercive, i.e., there exists a constant $\alpha \geq 0$ such that

$$a(u,u) \geq \alpha\|u\|^2 \quad \text{for all} \quad u \in E.$$

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In addition, we assume that

(iii) $b(u,v)$ is bounded, i.e., there exists a constant $\gamma \geq 0$ such that

$$|b(u,v)| \leq \gamma \|u\| \|v\| \quad \text{for all} \quad u,v \in E.$$

III. In theorems 3, 4, 5, we replace the continuity and bilinearity of $a(u,v)$ by the following:

(iv) $a(v, v-u) \geq a(u, v-u)$, for all $u,v \in E$.

(v) $a(u,v)$ is hemicontinuous in u, i.e., for every $u,v,p \in E$, the function

$$t \in [0,1] \to a(tu + (1-t)v, p)$$

is continuous.

(vi) $a(u,v)$ is convex and upper semicontinuous in v and $a(u,\theta) \geq 0$ for all $u \in E$ (θ is the zero element of E).

or

(vi′) $a(u,v)$ is linear upper semicontinuous in v.

It is obvious that condition (vi′) is a special case of (vi) and that if $a(u,v)$ is continuous bilinear and $a(u,u) \geq 0$, then (iv), (v), (vi), (vi′) hold.

2. Definitions and Lemma

Definition 1 *A set-valued map $T : X \to 2^{E'}$ is said to be hemi-lower-semicontinuous (hemi-l.s.c.) if for every fixed $u,v \in X$ the set-valued map*

$$t \in [0,1] \to T(tu + (1-t)v) \in 2^{E'}$$

is lower semicontinuous from [0,1] to $2^{E'}$ with respect to the weak topology of E'.*

Definition 1 could be phrased as follows: Given any sequence t_μ converging to t_0 on [0,1], denote $z_t = tu + (1-t)v$, and let $w_0 \in Tz_{t_0}$. Then there exists a sequence $w_\mu \in Tz_{t_\mu}$ which is weak* convergent to w_0.

Definition 2 *A set-valued map $T : X \to 2^{E'}$ is called*

(1°) Antimonotone, if for $u,v \in X, p \in Tu, q \in Tv$, we have

$$\langle p - q, u - v \rangle \leq 0;$$

(2°) Lipschitz map (when E is a Banach space), if there exists a constant ξ such that

$$H(Tu, Tv) \leq \xi \|u - v\| \quad \textit{for} \quad u,v \in X,$$

where H is a Hausdorff metric,

$$H(Tu,Tv) = \sup_{p \in E'} |d(p,Tu) - d(p,Tv)| = \max\{\sup_{p \in Tu} d(p,Tv), \sup_{p \in Tv} d(p,Tu)\}.$$

Lemma 1 *Let E be a Banach space and $X \subset E$ be a closed convex set. Suppose that $T : X \to 2^{E'}$ is a Lipschitz map and Tu is weakly compact in E' for each $u \in X$. Then T*

is hemi-l.s.c..

Proof For every fixed $u, v \in X$, let $z_t = tu + (1-t)v, t \in [0,1]$. Let $t_\mu \to t_0$ on [0,1] and $w_0 \in Tz_{t_0}$. Then

$$d(w_0, Tz_{t_\mu}) \le H(Tz_{t_0}, Tz_{t_\mu}) \le \xi\|z_{t_0} - z_{t_\mu}\| = \xi|t_\mu - t_0|\|u - v\|.$$

Since $Tz_{t\mu}$ is weakly compact, there exists $w_\mu \in Tz_{t\mu}$ such that

$$\|w_0 - w_\mu\| = d(w_0, Tz_{t\mu}) \le \xi|t_\mu - t_0|\|u - v\|.$$

Hence, $w_\mu \to w_0$ (strongly). Consequently, $w_\mu \to w_0$ (weak*ly). This is our required result.

3. Main results

We first use the Ky Fan fixed point theorem [3] to solve the Problem 1.

Lemma 2 *Let E be an l.c.s. and $X \subset E$ be a compact set. Suppose that $T : X \to 2^{E'}$ is upper semicontinuous and for each $u \in X, Tu$ is nonempty compact convex. If function $\varphi : X \times E' \times X \to R$ satisfies*

(1) for each fixed $v \in X, \varphi(u, w, v)$ is continuous in (u, w).

(2) for each fixed $(u, w) \in X \times E', \varphi(u, w, v)$ is lower semicontinuous and quasiconvex in v.

Then there exist $\bar{u} \in X, \bar{w} \in T\bar{u}$ such that

$$\varphi(\bar{u}, \bar{w}, v) \ge \varphi(\bar{u}, \bar{w}, \bar{u}) \quad \text{for all} \quad v \in X.$$

Proof We know that E' is an l.c.s. So $E \times E'$ is an l.c.s. Let a set-valued map $A : X \times E' \to X$ may be defined as follows:

$$A(u, w) = \{v \in X : \varphi(u, w, v) = \min_{s \in X} \varphi(u, w, s)\}.$$

Since φ is quasiconvex in $v, A(u, w)$ is convex. We shall verify that A is upper semicontinuous. In fact, let $(u_\mu, w_\mu) \to (u, w)$ on $X \times E'$ and $v_\mu \in A(u_\mu, w_\mu), v_\mu \to v$, then

$$\varphi(u_\mu, w_\mu, v_\mu) = \min_{s \in X} \varphi(u_\mu, w_\mu, s) \le \varphi(u_\mu, w_\mu, s) \quad \text{for all } s \in X.$$

By (1) and (2), we have

$$\varphi(u, w, v) \le \lim_\mu \varphi(u_\mu, w_\mu, v_\mu) \le \lim_\mu \varphi(u_\mu, w_\mu, s) = \varphi(u, w, s) \ \text{ for } s \in X.$$

By the compactness of $X, v \in X$ and $\varphi(u, w, v) = \min_{s \in X} \varphi(u, w, s)$. We have $v \in A(u, w)$. By [1, Corollary III. 1.9], A is upper semicontinuous. Since for $u \in X$, Tu is nonempty compact, $T(X)$ is a compact set. Hence $H = \overline{\text{Conv}}(T(X))$, the closed convex hull of $T(X)$, is a compact set. A set-valued map $S : X \times H \to 2^{X \times H}$ is defined as follows:

$S(u,w) = (A(u,w), Tu)$. It is easy to show that S is upper semicontinuous and $S(u,w)$ is nonempty compact convex. By Ky Fan theorem [2], there exists $(\bar{u},\bar{w}) \in X \times H$ such that $(\bar{u},\bar{w}) \in S(\bar{u},\bar{w})$. Consequently, $\bar{u} \in X, \bar{w} \in T\bar{u}$ such that

$$\varphi(\bar{u},\bar{w},v) \geq \varphi(\bar{u},\bar{w},\bar{u}) \quad \text{for all} \quad v \in X.$$

This completes the proof of Lemma 2.

Theorem 1 *Let E be an l.c.s. and $X \subset E$ be a nonempty compact convex set. Let $a(u,v)$ be a continuous bilinear form and $b(u,v)$ satisfy (i), (ii). Suppose that $T : X \to 2^{E'}$ is upper semicontinuous and for each $u \in X, Tu$ is a nonempty compact convex set. Then there exists a solution to Problem 1.*

Proof Let $\varphi(u,w,v) = a(u,v-u) + b(u,v) - \langle w, v-u\rangle$. It is easy to verify that φ satisfies requirements of Lemma 2. Hence there exist $\bar{u} \in X, \bar{w} \in T\bar{u}$ such that $\varphi(\bar{u},\bar{w},v) \geq \varphi(\bar{u},\bar{w},\bar{u})$ for $v \in X$. Consequently $(\bar{u},\bar{w})$ satisfies (1). This is the required result.

The following theorem is a noncompact form of Theorem 1. We note that the weak topology and weak* topology is consistent in a reflexive Banach space.

Theorem 2 *Let E be a reflexive Banach space and $X \subset E$ be a nonempty closed convex set. Let $a(u,v)$ be a coercive continuous bilinear form and $b(u,v)$ satisfy (i'), (ii), (iii). Suppose that $T : X \to 2^{E'}$ is upper semicontinuous from the weak topology $\sigma(E,E')$ of E to the weak topology $\sigma(E',E)$ of E' and for each $u \in X, Tu$ is nonempty bounded closed convex. If T is also a Lipschitz map and $\gamma + \xi < \alpha$, then Problem 1 has a solution.*

Proof We choose an arbitrarily fixed point $v_0 \in X$. For $u \in X, w \in Tu$, by the bilinearity, continuity, coercivity of $a(u,v)$ and (iii), we have

$$\begin{aligned} & a(u,v_0-u) + b(u,v_0) - b(u,u) - \langle w, v_0-u\rangle \\ &= a(u,v_0) - a(u,u) + b(u,v_0) - b(u,u) - \langle w, v_0-u\rangle \\ &\leq \beta\|u\|\|v_0\| - \alpha\|u\|^2 + \gamma\|u\|\|v_0\| + \gamma\|u\|^2 + \|w\|(\|v_0\| + \|u\|). \end{aligned} \tag{3}$$

Now since T is Lipschitz map, we have

$$d(w, Tv_0) \leq H(Tu, Tv_0) \leq \xi\|u - v_0\| \leq \xi(\|u\| + \|v_0\|).$$

Since Tv_0 is bounded closed convex, Tv_0 is weakly compact and there exists $p \in Tv_0$ such that $\|w-p\| = d(w,Tv_0)$. Hence we have

$$\|w\| \leq d(w,Tv_0) + \|p\| \leq \xi(\|u\| + \|v_0\|) + \max_{p \in Tv_0}\|p\|.$$

Thus, from inequality (3) we obtain

$$\begin{aligned} & a(u,v_0-u) + b(u,v_0) - b(u,u) - \langle w, v_0-u\rangle \\ &\leq -(\alpha - (\gamma+\xi))\|u\|^2 + (\beta+\gamma+2\xi)\|u\|\|v_0\| + (\|u\| + \|v_0\|)\max_{p\in Tv_0}\|p\| + \xi\|v_0\|^2. \end{aligned}$$

It is clear that for $\|u\|$ large enough, one has

$$a(u,v_0-u) + b(u,v_0) - b(u,u) - \langle w, v_0-u\rangle \leq 0. \tag{4}$$

Choose $k > 0$ large enough so that $\|v_0\| < k$, $\|u\| \geq k$ and inequality (4) holds. Let $B = \{u \in X : \|u\| \leq k\}$. Since E is a reflexive Banach space, B is a weakly compact convex subset of X. Note that a convex lower semicontinuous function is a weakly lower semicontinuous function and a linear continuous function is a weakly continuous function [13]. So $a(u,v)$ is a weakly continuous bilinear form and $b(u,v)$ is weakly continuous in u, weakly lower semicontiuous in v. Hence all requirements of Theorem 1 are satisfied in the weak topology on B. Thus, there exist $\bar{u} \in B, \bar{w} \in T\bar{u}$, such that

$$a(\bar{u}, v - \bar{u}) + b(\bar{u}, v) - b(\bar{u}, \bar{u}) \geq \langle \bar{w}, v - \bar{u} \rangle \quad \text{for all } v \in B. \tag{5}$$

Firstly, if $\|\bar{u}\| = k$ then since $v_0 \in B$, we have by (4), (5)

$$a(\bar{u}, v_0 - \bar{u}) + b(\bar{u}, v_0) - b(\bar{u}, \bar{u}) - \langle \bar{w}, v_0 - \bar{u} \rangle = 0. \tag{6}$$

Now for $v \in X$, choose $0 < t < 1$ small enough so that $v_t = tv + (1-t)v_0 \in B$. Use the convexity of b, linearity of a and (5), (6), we obtain

$$\begin{aligned} 0 \quad &\leq \quad a(\bar{u}, v_t - \bar{u}) + b(\bar{u}, v_t) - b(\bar{u}, \bar{u}) - \langle \bar{w}, v_t - \bar{u} \rangle \\ &\leq \quad t[a(\bar{u}, v - \bar{u}) + b(\bar{u}, v) - b(\bar{u}, \bar{u}) - \langle \bar{w}, v - \bar{u} \rangle \\ &\qquad + (1-t)[a(\bar{u}, v_0 - \bar{u}) + b(\bar{u}, v_0) - b(\bar{u}, \bar{u}) - \langle \bar{w}, v - \bar{u} \rangle \\ &= \quad t[a(\bar{u}, v - \bar{u}) + b(\bar{u}, v) - b(\bar{u}, \bar{u}) - \langle \bar{w}, v - \bar{u} \rangle. \end{aligned}$$

Therefore,

$$a(\bar{u}, v - \bar{u}) + b(\bar{u}, v) - b(\bar{u}, \bar{u}) \geq \langle \bar{w}, v - \bar{u} \rangle.$$

Secondly, if $\|\bar{u}\| < k$, then for $v \in X$, choose $0 < t < 1$ small enough so that $v_t = tv + (1-t)\bar{u} \in B$. By the demonstration in first step, we can show that $(\bar{u}, \bar{w})$ satisfies (1).

This completes the proof of the theorem.

Next, we shall use the technique of KKM and Ky Fan theorem [3] to solve Problem 2.

Theorem 3 *Let E be a t.v.s. and $X \subset E$ be a nonempty compact convex set. Suppose that $a(u,v)$ satisfies (iv), (v), (vi) and $b(u,v)$ satisfies (i), (ii). If $T : X \to 2^{E'}$ is antimonotone, hemi-l.s.c. and for $u \in X, Tu$ is nonempty, then there exists $\bar{u} \in X$ such that (2) holds.*

Proof Define a set-valued map $G : X \to 2^X$ as follows:

$$G(v) = \{u \in X : a(u, v - u) + b(u, v) - b(u, u) \geq \inf_{w \in Tu} \langle w, v - u \rangle\} \text{ for } v \in X.$$

We shall verify that G is KKM. Let $\{v_1, v_2, \cdots, v_n\} \subset X$, $u_0 \in \mathrm{Conv}\{v_1, v_2, \cdots, v_n\}$, i.e., there exist $t_i \in [0,1]$ $(i = 1, 2, \ldots, n)$ such that $\sum_{i=1}^{n} t_i = 1$, $u_0 = \sum_{i=1}^{n} t_i v_i$. If $u_0 \notin \bigcup_{i=1}^{n} G(v_i)$ then

$$a(u_0, v_i - u_0) + b(u_0, v_i) - b(u_0, u_0) < \inf_{w \in Tu_0} \langle w, v_i - u_0 \rangle, \quad i = 1, 2, \ldots, n.$$

By the convexity of a and b, we get

$$a(u_0, u_0 - u_0) + b(u_0, u_0) - b(u_0, u_0) < \inf_{w \in Tu_0} \langle w, u_0 - u_0 \rangle$$

which contradict the fact of $a(u_0, \theta) \geq 0$. This cotradiction shows that G is KKM.

Moreover we difine a set-valued map $\Gamma : X \to 2^X$ as follows

$$\Gamma(v) = \{u \in X : a(v, v-u) + b(u, v) - b(u, u) \geq \sup_{w \in Tv} \langle w, v-u \rangle\} \text{ for } v \in X.$$

Since T is antimonotone, we have

$$\inf_{w \in Tu} \langle w, v-u \rangle \geq \sup_{w \in Tv} \langle w, v-u \rangle. \tag{7}$$

By (iv) and (7), we obtain that for $v \in X, G(v) \subset \Gamma(v)$. Hence Γ is KKM. It is easy to verify that for $v \in X, \Gamma(v)$ is a closed set. As a result of Ky Fan theorem [3], we get $\bigcap_{v \in X} \Gamma(v) \neq \emptyset$.

Let $\bar{u} \in \bigcap_{v \in X} \Gamma(v)$ that is

$$a(v, v-\bar{u}) + b(\bar{u}, v) - b(\bar{u}, \bar{u}) \geq \sup_{w \in Tv} \langle w, v-\bar{u} \rangle \text{ for all } v \in X. \tag{8}$$

For each fixed $v \in X$, let $z_t = tv + (1-t)\bar{u} \in X,\ t \in [0,1]$. From (8), we have

$$a(z_t, z_t - \bar{u}) + b(\bar{u}, z_t) - b(\bar{u}, \bar{u}) \geq \sup_{w \in T_{z_t}} \langle w, z_t - \bar{u} \rangle \quad t \in [0,1].$$

By the convexity,

$$ta(z_t, v-\bar{u}) + t(b(\bar{u}, v) - b(\bar{u}, \bar{u})) \geq t \sup_{w \in Tz_t} \langle w, v-\bar{u} \rangle \quad t \in [0,1].$$

$$a(z_t, v-\bar{u}) + b(\bar{u}, v) - b(\bar{u}, \bar{u}) \geq \sup_{w \in T_{z_t}} \langle w, v-\bar{u} \rangle \quad t \in (0,1]. \tag{9}$$

Let $\bar{w} \in T\bar{u}$ be arbitrarily fixed. For arbitrary $\varepsilon > 0$, let

$$U_\varepsilon = \{w \in E' : |\langle w - \bar{w}, v - \bar{u} \rangle| < \varepsilon\}.$$

Then U_ε is a weak* open neighbourhood of $\bar{w}$ and $U_\varepsilon \cap T\bar{u} \neq \emptyset$. Since T is hemi-l.s.c. and $z_0 = \bar{u}$, there exists $\delta \in (0,1)$ such that as $t \in (0, \delta)$, $U_\varepsilon \cap Tz_t \neq \emptyset$. Let $w \in U_\varepsilon \cap Tz_t$. Then $w \in Tz_t$ and $|\langle w - \bar{w}, v - \bar{u} \rangle| < \varepsilon$. Therefore, $\langle w, v - \bar{u} \rangle > \langle \bar{w}, v - \bar{u} \rangle - \varepsilon$. Consequently,

$$\sup_{w \in Tz_t} \langle w, v - \bar{u} \rangle > \langle \bar{w}, v - \bar{u} \rangle - \varepsilon.$$

By the arbitrarity of ε and $\bar{w} \in T\bar{u}$, we have

$$\sup_{w \in Tz_t} \langle w, v - \bar{u} \rangle \geq \sup_{w \in T\bar{u}} \langle w, v - \bar{u} \rangle.$$

From (9), we obtain

$$a(z_t, v - \bar{u}) + b(\bar{u}, v) - b(\bar{u}, \bar{u}) \geq \sup_{w \in T\bar{u}} \langle w, v - \bar{u} \rangle \quad t \in (0, 1]. \tag{10}$$

By the hemicontinuity (v) of a, let $t \to 0$ in (10). Then

$$a(\bar{u}, v - \bar{u}) + b(\bar{u}, v) - b(\bar{u}, \bar{u}) \geq \sup_{w \in T\bar{u}} \langle w, v - \bar{u} \rangle$$

which is the required result.

According to Lemma 1, Theorem 3 and the proof of Theorem 2, we can obtain the following two theorems.

Theorem 4 *Let E be a reflexive Banach space and $X \subset E$ be a nonempty bounded closed convex set. Let $a(u,v)$ satisfy (iv), (v) (vi') and $b(u,v)$ satisfy (i'), (ii). Suppose that $T : X \to 2^{E'}$ is a antimonotone and Lipschitz map and for each $u \in X, Tu$ is a nonempty bounded closed convex set. Then there exists a solution of Problem 2.*

Theorem 5 *Let E be a reflexive Banach space and $X \subset E$ be a nonempty closed convex set. Let $a(u,v)$ be coercive and satisfy (iv), (v) (vi) and $b(u,v)$ satisfy (i'), (ii), (iii). Suppose that $T : X \to 2^{E'}$ is an antimonotone and Lipschitz map and for each $u \in X, Tu$ is a nonempty bounded convex closed set. If $\gamma + \xi < \alpha$, then there exists a solution to Problem 2.*

Corollary Let E, X be the same as in Theorem 5. Let $A : X \to E'$ be a hemicontinuous and monotone operator and $j : X \to R$ be convex lower semicontinuous. There exists $v_0 \in X$ such that A, j satisfy the following coercive condition:

$$\frac{\langle Au, u - v_0 \rangle + j(u)}{\|u\|} \to +\infty \quad \text{as} \quad \|u\| \to +\infty. \tag{11}$$

Suppose that $T : X \to 2^{E'}$ is an antimonotone and Lipachitz map and for each $u \in X, Tu$ is a nonempty bounded closed convex set. If $T(X)$ is bounded, then there exists a solution to Problem 2 with $a(u,v) = \langle Au, v \rangle$ and $b(u,v) = j(v)$.

Proof Let $a(u, v - u) = \langle Au, v - u \rangle$, $b(u,v) = j(v)$. Since $T(X)$ is bounded, there exists a constant $\varsigma > 0$ such that

$$\|w\| \leq \varsigma \quad \text{for all} \quad u \in X, \quad w \in Tu.$$

Then for $u \in X, w \in Tu$,

$$\begin{aligned} &\langle Au, v_0 - u \rangle + j(v_0) - j(u) - \langle w, v_0 - u \rangle \\ &\quad \leq -(\langle Au, u - v_0 \rangle + j(u)) + j(v_0) + \|w\| \|v_0 - u\| \\ &\quad \leq -(\langle Au, u - v_0 \rangle + j(u)) + \varsigma \|u\| + j(v_0) + \varsigma \|v_0\|. \end{aligned}$$

From the coercivity (11), we can see that

$$\langle Au, v_0 - u \rangle + j(v_0) - j(u) - \langle w, v_0 - u \rangle \leq 0 \text{ as } \|u\| \text{ is large enough.}$$

According to the proof of Theorem 2 and Theorem 3, the conclusion of Corollary is true.

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集值映象的变分不等式

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摘 要

本文应用Ky Fan 定理和KKM 技巧在(局部凸)Hausdorff 拓扑向量空间及自反Banach空间上讨论了集值映象的变分不等式解的存在性。所讨论的问题比[6,7,10,11] 中讨论的更为广泛。