On the other hand, $\{\xi^\alpha\}_{|\alpha|=n}$ forms a basis of π_n. So, we have the expression

$$p^*(x) = \sum_{|\alpha|=n} c_\alpha \xi^\alpha \quad \text{for some} \quad c_\alpha.$$

But, $p^*(v_i) = 0$ implies $c_\alpha = 0$ for $\alpha = ne_i$, $i = 0, 1, \ldots, n$. Therefore,

$$p^*(x) = \sum_{|\alpha|=n, \alpha_i<n} c_\alpha \xi^\alpha.$$

Hence, $|p^*(x) - p^{*,e}(x)| \le \sum_{|\alpha|=n,\alpha_i<n} \xi^\alpha \le \frac{1}{n}$. Thus,

$$|f(x) - p^{*,e}(x)| \le |f(x) - p^*(x)| + |p^*(x) - p^{*,e}(x)| \le 2E_n(f) + \frac{1}{n},$$

the theorem is proved. □

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单纯形上Bernstein多项式的一些性质

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摘　要

设$B_m(f,\cdot)$为函数f在d维单纯形σ上的n阶Bernstein多项式，本文对$f\in C^r(\sigma)$及$f\in C^{r+2}(\sigma)$给出了f的各阶偏导数用$B_n(f,\cdot)$相应偏导数逼近的误差估计. 同时也考虑了整系数Bernstein多项式的L_p模估计.

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Some Properties of Bernstein Polynomials on a Simples *

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Abstract Let $B_n(f,\cdot)$ be the Bernstein polynomial of degree n for a continuous function f with respect to a d-dimensional simplex σ. In this paper, the approximation error of partial derivatives of f by the partial derivatives of $B_n(f,\cdot)$ for $f \in C^r(\sigma)$ and for $f \in C^{r+2}(\sigma)$ are obtained. Also, the approximation, in L_p- norm, by the partial derivatives of Bernstein polynomials with integral coefficients on the simplex is considered.

Key words Bernstein polynomial, integral coefficient, derivative approximation.

1. Introduction

As usual, let $\mathbf{R}$ denote the set of all real numbers and $\mathbf{Z}_+$ the set of nonnegative integers. Let $\mathbf{N} := \mathbf{Z}_+\backslash\{0\}$. Thus $\mathbf{R}^d$ denotes the d-dimensional Euclidean space, and $\mathbf{Z}_+^d$ is a set of multi-index. Let σ be a d-dimensional simplex with vertex $v^0,\cdots,v^d$, here we assume that $v^i \in \mathbf{R}^d$, $i = 0,\cdots,d$ are in general positions; i.e., the vectors $v^i - v^0$, $i = 1,\cdots,d$ are linearly independent. It is clear that, for any $x \in \mathbf{R}^d$, there exists a unique vector $\xi = (\xi_0,\cdots,\xi_d) \in \mathbf{R}^{d+1}$ such that

$$x = \sum_{i=0}^{d} \xi_i v^i, \quad \sum_{i=0}^{d} \xi_i = 1.$$

The coefficients of $\xi = (\xi_0,\cdots,\xi_d)$ are called the barycentric coordinates of x with respect to the simplex σ. For $x = (x_1,\cdots,x_d) \in \mathbf{R}^d$, $y = (y_1,\cdots,y_d) \in \mathbf{R}^d$, $x\cdot y$ denotes the inner product of x and y, i.e.,

$$x \cdot y = \sum_{i=0}^{d} x_i y_i.$$

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Let $\alpha = (\alpha_0, \cdots, \alpha_d) \in \mathbf{Z}_+^{d+1}$ be a multi-index with the length $|\alpha| = \sum_{i=0}^{d} \alpha_i = n$, and $x \in \sigma$. The Bernstein polynomial basis of degree n is given by

$$B_\alpha(x) = \binom{n}{\alpha} \xi^\alpha$$

with $\binom{n}{\alpha} = \frac{n!}{\alpha_0!\alpha_1!\cdots\alpha_d!}$ and $\xi^\alpha = \xi^{\alpha_0}\xi^{\alpha_1}\cdots\xi^{\alpha_d}$. Clearly,

$$B_\alpha(\cdot) \geq 0, \text{ and } \sum_{|\alpha|=n} B_\alpha(\cdot) = 1$$

on σ. Associated with a continuous function $f \in C(\sigma)$, the n^{th} degree Bernstein polynomial of f with respect to σ is defined by

$$B_n(f, \cdot) = \sum_{|\alpha|=n} f(x_\alpha) B_\alpha(\cdot),$$

where the points $x_\alpha = \frac{1}{n}\sum_{i=0}^{d} \alpha_i v^i$ with $|\alpha| = n$ are called B-net points.

To consider the derivatives of functions defined over an arbitrary simplex, it is convenient to make use of directional derivatives. For $u, v \in \mathbf{R}^d$, let $y = u - v$, then the directional derivative of a function f with respect to y is defined as usual:

$$D_y f(\cdot) = \lim_{t \to 0} \frac{f(\cdot + ty) - f(\cdot)}{t} = \sum_{i=1}^{d} y_i \frac{\partial}{\partial x_i} f(\cdot), \quad f \in C^1(\mathbf{R}^d).$$

For convenience, corresponding to the barycentric coordinates of vertices $v^0, \cdots, v^d$, we use $e_0, \cdots, e_d$ to denote the unit vectors in $\mathbf{R}^{d+1}$. The directional derivatives with respect to the directions $v^i - v^0$, $i = 1, \ldots, d$, or $e_i - e_0$, $i = 1, \ldots, d$ in barycentric coordinates, are denoted by D_i, $i = 1, \ldots, d$. If we identify σ with the d- dimensional standard simplex s_d, then the directional derivatives D_i coincide with the partial derivatives $\frac{\partial}{\partial x_i}$. As a consequence, we can replace multiple partial derivatives of a function f on σ by

$$D^\beta f(\cdot) = (D_1^{\beta_1} \cdots D_d^{\beta_d} f(\cdot), \quad f \in C^{|\beta|}(\sigma),$$

where $\beta = (\beta_1, \cdots, \beta_d) \in \mathbf{Z}_+^d$.

The main purpose of this paper is to investigate the approximation properties of the derivatives of $B_n(f, \cdot)$ and $B_n^e(f, \cdot)$, the Bernstein polynomial with integral coefficients. The paper is organized as follows. In Section 2, we estimate the errors of partial derivatives $D^\beta f$ of $f \in C^r(\sigma)$ and $f \in C^{r+2}(\sigma)$ approximated by $D^\beta B_n(f, \cdot)$ with $|\beta| \leq r$ respectively. Section 3, we discuss the derivative approxmation, in L_p-norm, of Bernstein polynomials on σ with integral coefficients.

2. Derivative approximation on $B_n(f,\cdot)$

For the simplex $\sigma=[V]=[v^0,\cdots,v^d]$, the boundary of σ is made up of faces, i.e., of convex hulls of subsets of $V=\{v^0,\cdots,v^d\}$. For any $W\subset V$, we call $[W]$ the W-face of σ. If the center point of the circumscribed sphere of the W-face is inside of $[W]$, we call $[W]$ the central side face. For any simplex σ, there exists one of its faces which should be central side face. Let $O(W)$ denote the circumscribed sphere of W, and ρ_w the radius of $O(W)$. We define

$$\rho=\max\{\rho_w; W-\text{face is central side face of }\sigma\}. \tag{1}$$

For $x\in\mathbf{R}^d$, $\xi=(\xi_0,\xi_1,\cdots,\xi_d)$ are barycentric coordinates of x with respect to $\sigma=[V]$. Let

$$h(x)=\sum_{i=0}^{d}\xi_i v^i\cdot v^i-\sum_{i=0}^{d}\sum_{j=0}^{d}\xi_i\xi_j v^i\cdot v^j.$$

It is easy to see that $h(v^i)=0$, for $i=0,1,\cdots,d$. Notice that $x_\alpha-x=\sum_{i=0}^{d}(\frac{\alpha_i}{n}-\xi_i)v^i$, we can easily figure out that

$$\sum_{|\alpha|=n}\|x_\alpha-x\|^2B_\alpha(x)=\frac{h(x)}{n}. \tag{2}$$

Furthermore, Jia and Wu [JW] point out that

$$\max_{x\in\sigma}h(x)=\rho^2. \tag{3}$$

The Bernstein polynomial provides an approximation to $f\in C(\sigma)$, which, on σ, converges uniformly to f as $n\to\infty$. For the functions f with continuous partial derivatives, let $f'=(\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_d})$ and $\omega_1(\delta)=\max_{\|y\|\le\delta}\|f'(\cdot,+y)-f'(\cdot)\|$. Then the following estimation holds. In particularly, if $d=1$, we get the theorem 1.6.2 in $[L]$.

Theorem 2.1 *For $f\in C^1(\sigma)$, then*

$$|f(x)-B_n(f,x)|\le(\rho+\rho^2)\frac{1}{\sqrt{n}}\omega_1(\frac{1}{\sqrt{n}}),\quad x\in\sigma, \tag{4}$$

where ρ is given by (1).

Proof By the mean value theorem,

$$\begin{aligned} f(x)-f(y) &= D_{x-y}f(z)=f'(z)\cdot(x-y)\\ &= f'(y)\cdot(x-y)+(f'(z)-f'(y))\cdot(x-y), \end{aligned}$$

where $z = y + \theta(x - y)$ and $0 < \theta < 1$. Note that the absolute value of the last term does not exceed $\|x - y\|(1 + \frac{\|x - y\|}{\delta})\omega_1(\delta)$. One gets that

$$\begin{aligned}|f(x) - B_n(f,x)| &= |\sum_{|\alpha|=n}(f(x) - f(x_\alpha))B_\alpha(x)| \\ &\le |\sum_{|\alpha|=n}(f'(x)\cdot(x - x_\alpha)B_\alpha(x)| + \sum_{|\alpha|=n}\|f'(z) - f'(x)\|\|x - x_\alpha\|B_\alpha(x).\end{aligned}$$

The first sum becomes zero since $B_\alpha(x,x) = x$. Using Schwarz inequality and (2), and choose $\delta = \frac{1}{\sqrt{n}}$, we have

$$\begin{aligned}|f(x) - B_n(f,x)| &\le \omega_1(\delta)\sum_{|\alpha|=n}(\|x - x_\alpha\|B_\alpha(x) + \frac{\|x - x_\alpha\|^2}{\delta}B_\alpha(x)) \\ &\le \omega_1(\delta)((\sum_{|\alpha|=n}\|x - x_\alpha\|^2 B_\alpha(x))^{1/2} + \frac{1}{\delta}\sum_{|\alpha|=n}\|x - x_\alpha\|^2 B_\alpha(x)) \\ &\le \omega_1(\delta)((\frac{h(x)}{n})^{1/2} + \frac{h(x)}{n\delta}) \\ &= \frac{1}{\sqrt{n}}\omega_1(\frac{1}{\sqrt{n}})(\sqrt{h(x)} + h(x)).\end{aligned}$$

So, we obtain the pointwise error estimation

$$|f(x) - B_n(f,x)| \le \frac{1}{\sqrt{n}}\omega_1(\frac{1}{\sqrt{n}})(\sqrt{h(x)} + h(x)).$$

Combine it with (3), the proof is complete. □

The following exact error estimate for $f \in C^2(\sigma)$ approximated by Bernstein polynomial $B_n(f,\cdot)$ is due to Jia and Wu [JW].

Theorem A *Let $f \in C^2(\sigma)$ and $M = \max_{1\le i,j\le d}\|D_iD_jf\|_\infty$. Then*

$$|f(x) - B_n(f,x)| \le \frac{dM\rho^2}{2n}, \quad x \in \sigma,$$

where ρ is defined by (1) and the coefficient before $\frac{1}{n}$ is sharp.

To estimate derivative approximation, we need the forward difference operator which is defined inductively as follwos:

$$\begin{aligned}\Delta_i^0 f(x_\alpha) &= f(x_\alpha), \\ \Delta_i^k f(x_\alpha) &= \Delta_i^{k-1} f(x_{\alpha+e_i-e_0}) - \Delta_i^{k-1} f(x_\alpha), \\ \Delta^\beta f(x_\alpha) &= \Delta_1^{\beta_1}\Delta_2^{\beta_2}\cdots\Delta_d^{\beta_d} f(x_\alpha) \quad \text{for } \beta \in \mathbf{Z}_+^d.\end{aligned}$$

It is not difficult to show that the following derivative formula for Bernstein polynomial holds:

$$D^\beta B_n(f,x) = \frac{n!}{(n - |\beta|)!}\sum_{|\alpha|=n-|\beta|}\Delta^\beta f(x_{\alpha+|\beta|e_0})B_\alpha(x). \tag{5}$$

In fact, it suffices to show that

$$D_i^k B_n(f,x) = \frac{n!}{(n-k)!} \sum_{|\alpha|=n-k} \Delta_i^k f(x_{\alpha+ke_0}) B_\alpha(x). \tag{6}$$

Denote $z = v^i - v^0$. Then

$$\begin{aligned} D_i B_\alpha(x) &= \lim_{t\to 0} \frac{B_\alpha(x+tz) - B_\alpha(x)}{t} \\ &= \lim_{t\to 0} \frac{\begin{pmatrix} n \\ \alpha \end{pmatrix} ((\xi + te_i - te_0)^\alpha - \xi^\alpha)}{t} \\ &= n!(\frac{1}{(\alpha - e_i)!}\xi^{\alpha-e_i} - \frac{1}{(\alpha-e_0)!}\xi^{\alpha-e_0}) \\ &= n(B_{\alpha-e_i}(x) - B_{\alpha-e_0}(x)). \end{aligned}$$

Hence,

$$D_i B_\alpha(f,x) = n \sum_{|\alpha|=n} f(x_\alpha)(B_{\alpha-e_i}(x) - B_{\alpha-e_0}(x)) = n \sum_{|\alpha|=n-1} \Delta_i f(x_{\alpha+e_0}) B_\alpha(x),$$

and (6) follows by induction.

Now, we are in a position to prove the following

Theorem 2.2 *If $f \in C^r(\sigma)$, and $\omega(f,\delta)$ is the modulus of continuity of function f, then for any $\beta \in \mathbf{Z}_+^d$, $|\beta| \le r$, we have*

$$\begin{aligned} |D^\beta f(x) - D^\beta B_n(f,x)| &\le (2 + A + \frac{|\beta|}{\sqrt{n-|\beta|}})\omega(D^\beta f, \frac{1}{\sqrt{n-|\beta|}}) \\ &\quad + \frac{|\beta|(|\beta|-1)}{2n}\|D^\beta f\|_\infty \end{aligned} \tag{7}$$

on σ, where $A = \min\{\rho, \rho^2\}$.

Proof Let $B^\beta_{n-|\beta|}(f,\cdot) := \frac{n^{|\beta|}(n-|\beta|)!}{n!} D^\beta B_n(f,\cdot)$. Then

$$\begin{aligned} &|D^\beta f(\cdot) - D^\beta B_n(f,\cdot)| \le |D^\beta f(\cdot) - B_{n-|\beta|}(D^\beta f,\cdot)| \\ &\quad + |B_{n-|\beta|}(D^\beta f,\cdot) - B^\beta_{n-|\beta|}(f,\cdot)| + |B^\beta_{n-|\beta|}(f,\cdot) - D^\beta B_{n-|\beta|}(f,\cdot)|. \end{aligned} \tag{8}$$

By (2), (3) and using standard inequality technique, one sees that, the first term does not exceed

$$(1+A)\omega(D^\beta f, \frac{1}{\sqrt{n-|\beta|}}). \tag{9}$$

On the other hand, by the mean value theorem, we have $\Delta^\beta f(x_{\alpha+|\beta|e_0}) = \frac{1}{n^{|\beta|}} D^\beta f(z_\alpha)$ for some z_α. Combine with (5) and notice that $\|z_\alpha - x_\alpha\| \le \frac{|\beta|}{n-|\beta|}$, the second term becomes

$$\begin{aligned}
&|\sum_{|\alpha|=n-|\beta|}(D^\beta f(x_\alpha)-D^\beta f(z_\alpha))B_\alpha(x)| \\
&\le \sum_{|\alpha|=n-|\beta|}(1+\frac{\|z_\alpha-x_\alpha\|}{\delta})\omega(D^\beta f,\delta)B_\alpha(x) \\
&\le (1+\frac{|\beta|}{\sqrt{n-|\beta|}})\omega(D^\beta f,\frac{1}{\sqrt{n-|\beta|}}), \qquad (10)
\end{aligned}$$

the last inequality is obtained by choosing $\delta=\frac{1}{\sqrt{n-|\beta|}}$. Furthermore,

$$\prod_{i=1}^{k}(1-x_i)\ge 1-\sum_{i=1}^{k}x_i$$

holds for all $x_i\in[0,1]$, therefore,

$$\begin{aligned}
0<1-\frac{n(n-1)\cdots(n-k+1)}{n^k} &= 1-(1-\frac{1}{n})\cdots(1-\frac{k-1}{n}) \\
&\le \sum_{i=1}^{k-1}\frac{i}{n}=\frac{k(k-1)}{2n}.
\end{aligned}$$

Thus, by the derivative formula, the third term is going to be

$$\begin{aligned}
|(1-\frac{(n-|\beta|)!n^{|\beta|}}{n!})D^\beta B_n(f,x)| &= |(\frac{n!}{(n-|\beta|)!n^{|\beta|}}-1)\sum_{|\alpha|=n-|\beta|}D^\beta f(z_\alpha)B_\alpha(x)| \\
&\le (1-\frac{n(n-1)\cdots(n-|\beta|+1)}{n^{|\beta|}})\|D^\beta f\|_\infty \\
&\le \frac{|\beta|(|\beta|-1)}{2n}\|D^\beta f\|_\infty. \qquad (11)
\end{aligned}$$

Combine (9) to (11) with (8), the theorem is proved. □

For the usual univariate forward difference operator Δ : $\Delta f(x)=f(x+1)-f(x)$, the following facts are well-known:

$$\Delta^n x^m=\begin{cases}0, & \text{if } 0\le m<n\\ n!, & \text{if } m=n\end{cases} \qquad (12)$$

By induction, we can prove that

$$\Delta^n x^{n+1}=(n+1)!(x+\frac{n}{2}) \qquad (13)$$

and

$$\Delta^n x^{n+2}=\frac{(n+2)!}{2}(x^2+nx+\frac{1}{12}n(3n+1)). \qquad (14)$$

For any direction vector $z = \sum_{i=0}^{d} \varsigma_i v^i$ with $\sum_{i=0}^{d} \varsigma_i = 0$, we have

$$\begin{aligned} D_z f(u) &= \lim_{t\to 0} \frac{f(u + t\sum_{i=0}^d \varsigma_i v^i) - f(u)}{t} \\ &= \lim_{t\to 0} \frac{f(u + t\sum_{i=0}^d \varsigma_i (v^i - v^0)) - f(u)}{t} \\ &= \sum_{i=1}^{d} \varsigma_i D_i f(u) = \sum_{i=1}^{d} \varsigma_i \cdot e^i D_i f(u), \end{aligned}$$

here, we use e^i, $i = 1, \ldots, d$ to denote the unit coordinate vectors in $\mathbf{R}^d$ to distinguish $e_i \in \mathbf{R}^{d+1}$.

Now, for $f \in C^{r+2}(\sigma)$, we prove the following result.

Theorem 2.3 *Suppose that* $f \in C^{r+2}(\sigma)$ *and* $\beta \in \mathbf{Z}_+^d$. *Let* $M_1 = \max_{1\le i\le d} \|D^{\beta+e_i} f\|_\infty$ *and* $M_1 = \max_{1\le i,j\le d} \|D^{\beta+e_i+e_j} f\|_\infty$. *Then for any* $|\beta| \le r$

$$\begin{aligned} |D^\beta f(x) - B^\beta_{n-|\beta|}(f,x)| \le\ & \frac{dM\rho^2}{2(n-|\beta|)} + |\beta| M_1 (\frac{1}{2n} + \frac{|\alpha|}{n-|\beta|}) \\ & + |\beta|^2 M_2 (\frac{1}{2n} + \frac{|\alpha|}{n-|\beta|})^2 + \frac{|\beta| M_2}{24n^2}, \end{aligned}$$

where

$$B^\beta_{n-|\beta|}(f,x) := \frac{n^{|\beta|}(n-|\beta|)!}{n!} D^\beta B_n(f,x).$$

Proof Since

$$\begin{aligned} |D^\beta f(x) - B^\beta_{n-|\beta|}(f,x)| \le\ & |D^\beta f(x) - B_{n-|\beta|}(D^\beta f, x)| \\ & + |B_{n-|\beta|}(D^\beta f, x) - B^\beta_{n-|\beta|}(f,x)|, \end{aligned}$$

from Theorem A, the first term is bounded by

$$\frac{dM\rho^2}{2(n-|\beta|)}. \tag{15}$$

To estimate the second term, we use Taylor expansion formula

$$f(y) = \sum_{k=0}^{|\beta|+1} \frac{1}{k!} D^k_{y-x} f(x) + \frac{1}{(|\beta|+2)!} D^{|\beta|+2}_{y-x} f(x + \theta(y-x)),$$

where $0 < \theta < 1$.

Let $\xi = (\xi_0, \xi_1, \cdots, \xi_d)$ and $\varsigma = (\varsigma_0, \varsigma_1, \cdots, \varsigma_d)$ be the barycentric coordinates of x and y, and $\bar\xi = (\xi_1, \cdots, \xi_d)$, $\bar\varsigma = (\varsigma_1, \cdots, \varsigma_d)$. Then $D_{y-x} = \sum_{i=1}^{d} (\bar\xi - \bar\varsigma) \cdot e^i D_i$, so,

$$D^k_{y-x} = \left(\sum_{i=1}^{d} (\bar\xi - \bar\varsigma) \cdot e^i D_i\right)^k = \sum_{\eta\in \mathbf{Z}_+^d,\ |\eta| = k} \binom{k}{\eta} (\bar\xi - \bar\varsigma)^\beta D^\beta.$$

For $\beta,\gamma\in\mathbf{Z}_+^d$, write $\beta^*=(0,\beta_1,\cdots,\beta_d)$, $\gamma^*=(0,\gamma_1,\cdots,\gamma_d)$. Thus

$$\begin{aligned}\Delta^\beta f(x_{\alpha+|\beta|e_0}) &= \sum_{\gamma\le\beta}(-1)^{\beta-\gamma}\binom{\beta}{\gamma} f(x_{\alpha+\beta^*-\gamma^*+|\gamma|e_0})\\ &= \sum_{\gamma\le\beta}(-1)^{\beta-\gamma}\binom{\beta}{\gamma}\sum_{k=0}^{|\beta|+1}\frac{1}{k!}\Big(\sum_{i=1}^{d}\Big(\frac{\beta-\gamma}{n}-\frac{|\beta|\bar\alpha}{n-|\beta|}\Big)\cdot e_iD_i\Big)^k f(x_\alpha)\\ &\quad+\sum_{\gamma\le\beta}(-1)^{\beta-\gamma}\binom{\beta}{\gamma}\frac{1}{(|\beta|+2)!}\Big(\sum_{i=1}^{d}\Big(\frac{\beta-\gamma}{n}-\frac{|\beta|\bar\alpha}{n-|\beta|}\Big)\cdot e_iD_i\Big)^{|\beta|+2}f(z_\alpha),\qquad(16)\end{aligned}$$

where $z_\alpha=x_\alpha+\theta(x_{\alpha+\beta-\gamma+|\gamma|e_0}-x_\alpha)$, for some $0<\theta<1$, and $\bar\alpha=(\alpha_1,\cdots,\alpha-d)$ for $\alpha=(\alpha_1,\cdots,\alpha-d)$.

The first sum in (16) can be written as

$$\begin{aligned}&\sum_{k=0}^{|\beta|+1}\frac{1}{n^kk!}\sum_{|\eta|=k}\binom{k}{\eta}D^\eta f(x_\alpha)\sum_{\gamma\le\beta}(-1)^{\beta-\gamma}\binom{\beta}{\gamma}\Big(\beta-\gamma\quad\frac{n|\beta|\bar\alpha}{n-|\beta|}\Big)^\eta\\ &=\sum_{k=0}^{|\beta|+1}\frac{1}{n^kk!}\sum_{|\eta|=k}\binom{k}{\eta}D^\eta f(x_\alpha)\Delta^\beta(x-x_0)^\eta|_{x=0},\end{aligned}$$

where $x_0=\dfrac{n|\beta|}{n-|\beta|}\bar\alpha$. So, from (16), only the last two terms are nonzero. They are

$$I_1=\frac{1}{n^{|\beta|}|\beta|!}\sum_{|\eta|=|\beta|}\binom{|\beta|}{\eta}D^\eta f(x_\alpha)\Delta^\beta(x-x_0)^\eta|_{x=0}=\frac{1}{n^{|\beta|}}D^\beta f(x_\alpha),$$

and

$$\begin{aligned}I_2 &= \frac{1}{n^{|\beta|+1}(|\beta|+1)!}\sum_{|\eta|=|\beta|+1}\binom{|\beta|+1}{\eta}D^\eta f(x_\alpha)\Delta^\beta(x-x_0)^\eta|_{x=0}\\ &= \frac{1}{n^{|\beta|+1}(|\beta|+1)!}\sum_{i=1}^{d}\binom{|\beta|+1}{\beta+e_i}D^{\beta+e^i}f(x_\alpha)(\beta+e^i)!\Big(\frac{\beta_i}{2}-\frac{n|\beta|\alpha_i}{n-|\beta|}\Big)\\ &= \frac{1}{2n^{|\beta|+1}}\sum_{i=1}^{d}\Big(\beta_i-\frac{2n|\beta|\alpha_i}{n-|\beta|}\Big)D^{\beta+e^i}f(x_\alpha).\end{aligned}$$

The second equality in I_1 holds because only the term $\eta=\beta$ in the sum is nonzero and by (12). In I_2, we notice that only the terms for $\eta=\beta+e^i$ in the sum are nonzero and use (13) to get the second equality.

The second sum in (16)

$$\begin{aligned}I_3 &= \frac{1}{n^{|\beta|+2}(|\beta|+2)!}\sum_{|\eta|=|\beta|+2}\binom{|\beta|+2}{\eta}D^\eta f(z_\alpha)\Delta^\beta(x-x_0)^\eta|_{x=0}\\ &= \frac{1}{n^{|\beta|+2}(|\beta|+2)!}\sum_{i,j=1}^{d}\binom{|\beta|+2}{\beta+e^i+e^j}D^{\beta+e^i+e^j}f(z)\Delta^\beta(x-x_0)^{\beta+e^i+e^j}|_{x=0}.\end{aligned}$$

If $i \neq j$, apply the fact (13), it becomes

$$\frac{1}{4n^{|\beta|+2}}\sum_{i\neq j}(\beta_i - \frac{2n|\beta|\alpha_i}{n-|\beta|})(\beta_j - \frac{2n|\beta|\alpha_j}{n-|\beta|})D^{\beta+e^i+e^j}f(z_\alpha).$$

If $i = j$, using the fact (14), it will be

$$\frac{1}{2n^{|\beta|+2}}\sum_{i=1}^{d}((\frac{n|\beta|}{n-|\beta|})^2\alpha_i^2 - \frac{n|\beta|}{n-|\beta|}\alpha_i\beta_i + \frac{\beta_i(3\beta_i+1)}{12})D^{\beta+2e^i}f(z_\alpha).$$

Therefore,

$$\begin{aligned} I_3 &= \frac{1}{4n^{|\beta|+2}}\sum_{i\neq j}(\beta_i - \frac{2n|\beta|}{n-|\beta|}\alpha_i)(\beta_j - \frac{2n|\beta|}{n-|\beta|}\alpha_j)D^{\beta+e^i+e^j}f(z_\alpha) \\ &\quad + \frac{1}{2n^{|\beta|+2}}\sum_{i=1}((\frac{n|\beta|}{n-|\beta|})^2\alpha_i^2 - \frac{n|\beta|}{n-|\beta|}\alpha_i\beta_i + \frac{\beta_i(3\beta_i+1)}{12})D^{\beta+2e^i}f(z_\alpha). \end{aligned}$$

Using derivative formula and the above facts, we have

$$\begin{aligned} &|B_{n-|\beta|}(D^\beta f, x) - B^\beta_{n-|\beta|}(f,x)| = |\sum_{|\alpha|=n-|\beta|}(D^\beta f(x_\alpha) - n^{|\beta|}\Delta^\beta f(x_{\alpha+|\beta|e_0}))B_\alpha(x)| \\ &\leq |\beta|M_1(\frac{1}{2n} + \frac{|\alpha|}{n-|\beta|}) + |\beta|^2M_2(\frac{1}{2n} + \frac{|\alpha|}{n-|\beta|})^2 + \frac{|\beta|M_2}{24n^2}. \end{aligned}$$

Combine with (15), the conclusion follows.

We mention that if $|\beta| = 0$, then Theorem 2.3 will be Theorem A.

3. Bernstein polynomials with integral coefficients

Martinez [M] considered the derivative approximation using tensor product generalization of Bernstein polynomials with integral coefficients. In this section, we shall investigate Bernstein polynomials with integral coefficients on a simplex.

We use V_σ to denote the volume of the d-dimensional simplex $\sigma = [v^0, \cdots, v^d]$, for $v^i = (v^i_1, \cdots, v^i_d)$, i.e., $V_\sigma = \frac{1}{d!}|\det(v^i_j - v^0_j)^d_{i,j=1}|$.

Associated with Bernstein polynomial $B_n(f,\cdot)$, we define the Bernstein polynomial with integral coefficients as

$$B^e_n(f,x) = \sum_{|\alpha|=n}\left[f(x_\alpha)\begin{pmatrix} n \\ \alpha \end{pmatrix}\right]\xi^\alpha,$$

where $[\cdot]$ represents the greatest integer function. Corresponding to $D^\beta B_n(f,\cdot)$ and with the aid of derivative formula (5), we set

$$(D^\beta B_n)^e(f,\cdot) = \sum_{|\alpha|=n-|\beta|}\left[\frac{n!}{(n-|\beta|)!}\Delta^\beta f(x_{\alpha+|\beta|e_0})\right]B_\alpha(\cdot).$$

As usual, the L_p-norm of $f \in L_p(\sigma)$ is denoted by $\|f\|_p$.

We have the following L_p-norm estimation for the approximation of $(D^\beta B_n)^e(f,\cdot)$.

Theorem 3.1 *Let $f \in C^r(\sigma)$. Then for $1 \le p < \infty$, $x \in \sigma$,*

$$\|D^\beta f(x) - (D^\beta B_n)^e(f,x)\|_p \le V_\sigma^{1/p}[(2 + A + \frac{|\beta|}{\sqrt{n-|\beta|}})\omega(D^\beta f, \frac{1}{\sqrt{n-|\beta|}})$$
$$+ \frac{|\beta|(|\beta|-1)}{n}\|D^\beta f\|_\infty + \frac{1}{n-|\beta|}] + \left(\frac{d!V_\sigma}{((n-|\beta|)p+1)\cdots((n-|\beta|)p+d)}\right)^{1/p},$$

where $A = \min\{\rho, \rho^2\}$, and ρ is given by (1).

Proof Clearly, we have $0 \le B_n(f,x) - B_n^e(f,x) \le g_n(x)$ on σ, where $g_n(x) = \sum_{|\alpha|=n} \xi^\alpha$.

Since $\binom{n}{\alpha} \ge n$, if there exists $0 < \alpha_i < n$, one gets

$$g_n(x) = \sum_{|\alpha|=n} \xi^\alpha + \sum_{i=0}^{d} \xi_i^d \le \frac{1}{n} \sum_{|\alpha|=n,\ \exists\ 0<\alpha_i<n} \binom{n}{\alpha} \xi^\alpha + \sum_{i=0}^{d} \xi_i^n \le \frac{1}{n} + \sum_{i=0}^{d} \xi_i^n.$$

Hence,

$$\|g_n(x)\|_p \le \|\frac{1}{n}\|^p + \sum_{i=0}^{d} \|\xi_i^n\|_p = V_\sigma^{1/p}\frac{1}{n} + \left(\frac{d!V_\sigma}{(np+1)\cdots(np+d)}\right)^{1/p}$$

Here, we used the fact, $\|\xi_i\|_p^p = dV_\sigma B(np+1,d) = \frac{d!V_\sigma}{(np+1)\cdots(np+d)}$, which can be obtained by direct calculation and $B(p,q)$ is Beta-function. Therefore, using Theorem 2.2 and

$$\|D^\beta f(x) - (D^\beta B_n)^e(f,x)\|_p \le \|D^\beta f(x) - D^\beta B_n(f,x)\|_p + \|g_{n-|\beta|}(x)\|_p, \qquad (17)$$

the conclusion is obtained. □

For $p = \infty$, we prove.

Theorem 3.2 *Suppose that $f \in C^r(\sigma)$ and*

$$\frac{n!}{(n-|\beta|)!}\Delta^\beta f(x_{\alpha+|\beta|e_0}) \quad \textit{for} \quad \alpha_j = n - |\beta|, \quad j = 0, 1, \cdots, d$$

are integers, then

$$\|D^\beta f(x) - (D^\beta B_n)^e(f,x)\|_\infty \le (2 + \frac{|\beta|}{\sqrt{n-|\beta|}} + A)\omega(D^\beta f, \frac{1}{\sqrt{n-|\beta|}})$$
$$+ \frac{|\beta|(|\beta|-1)}{2n}\|D^\beta f\|_\infty + \frac{1}{n}. \qquad (18)$$

Proof By the assumption, we have

$$|D^\beta B_n(f,x) - (D^\beta B_n)^e(f,x)| \le g_{n-|\beta|}(x) \le \frac{1}{n},$$

with the aid of (17) and Theorem 2.2, (18) is obtained. □

Let $B^{\beta,e}_{n-|\beta|}(f,\cdot)$ denote the integral coefficient polynomial of $B^{\beta}_{n-|\beta|}(f,\cdot)$ which is defined in Theorem 2.3. Similarly, by using Theorem 2.3 we can establish the following

Theorem 3.3 *Let $f \in C^{r+2}(\sigma)$, $\beta \in \mathbf{Z}^d_+$ and $|\beta| \le r$.*

1) if $1 \le p < \infty$, then

$$\|D^\beta f(\cdot) - B^{\beta,e}_{n-|\beta|}(f,\cdot)\|_p \le V_\sigma^{1/p}\left[\frac{dM_2\rho^2+2}{2(n-|\beta|)} + |\beta|M_1(\frac{1}{2n}+\frac{|\alpha|}{n-|\beta|}) + |\beta|^2M_2(\frac{1}{2n}+\frac{|\alpha|}{n-|\beta|})^2 + \frac{|\beta|M_2}{24n^2}\right] + \left(\frac{d!V_\sigma}{((n-|\beta|)p+1)\cdots((n-|\beta|)p+d)}\right)^{1/p};$$

2) if $p=\infty$, and the numbers $n^{|\beta|}\Delta^\beta f(x_{\alpha+|\beta|e_0})$ for $\alpha_j = n-|\beta|$, $j=0,1,\ldots,d$ are integers, then

$$\begin{aligned}\|D^\beta f(\cdot) - B^{\beta,e}_{n-|\beta|}(f,\cdot)\|_\infty \le\ & \frac{dM_2\rho^2+2|\alpha||\beta|M-1}{2(n-|\beta|)} + \frac{|\beta|M_1+2}{2n} \\ & + |\beta|^2M_2(\frac{1}{2n}+\frac{|\alpha|}{n-|\beta|})^2 + \frac{|\beta|M_2}{24n^2}.\end{aligned}$$

As an application of Theorem 3.2, we have the following generalization of the Kantorovic theorem [K].

Corollary 3.1 *Let f be a continuous function on σ with $f(v_i)=0$, then*

$$E_{n,e}(f) \le 2E_n(f) + \frac{1}{n},$$

where $E_n(f) = \inf_{p\in\pi_n}\|f-p\|_\infty$, $E_{n,e}(f) = \inf_{p\in\pi_{n,e}}\|f-p\|_\infty$ and π_n the space of all polynomials of degree$\le n$; $\pi_{n,e}$ the space of all polynomials in π_n with integral coefficients.

Proof By the existence theorem, there is a polynomial $p \in \pi_n$ such that

$$\|f-p\|_\infty = E_n(f).$$

Since $f(v_i)=0$, we get $|p(v_i)| \le E_n(f)$. Now, let $L(x)$ be a linear function satisfies $L(v_i)=p(v_i)$, then $\|L(x)\| \le \max_i |L(v_i)| \le E_n(f)$. Write $p^* = p(x) - L(x)$, so $p^*(v_i)=0$ and

$$\begin{aligned}|f(x)-p^*(x)| &\le |f(x)-p(x)| + |p(x)-p^*(x)| \\ &\le E_n(f) + \|L(x)\|_\infty \le 2E_n(f).\end{aligned}$$

On the other hand, $\{\xi^\alpha\}_{|\alpha|=n}$ forms a basis of π_n. So, we have the expression

$$p^*(x) = \sum_{|\alpha|=n} c_\alpha \xi^\alpha \quad \text{for some} \quad c_\alpha.$$

But, $p^*(v_i) = 0$ implies $c_\alpha = 0$ for $\alpha = ne_i$, $i = 0, 1, \ldots, n$. Therefore,

$$p^*(x) = \sum_{|\alpha|=n, \alpha_i<n} c_\alpha \xi^\alpha.$$

Hence, $|p^*(x) - p^{*,e}(x)| \le \sum_{|\alpha|=n,\alpha_i<n} \xi^\alpha \le \frac{1}{n}$. Thus,

$$|f(x) - p^{*,e}(x)| \le |f(x) - p^*(x)| + |p^*(x) - p^{*,e}(x)| \le 2E_n(f) + \frac{1}{n},$$

the theorem is proved. □

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单纯形上Bernstein多项式的一些性质

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摘　要

设$B_m(f,\cdot)$为函数f在d维单纯形σ上的n阶Bernstein多项式，本文对$f \in C^r(\sigma)$及$f \in C^{r+2}(\sigma)$给出了f的各阶偏导数用$B_n(f,\cdot)$相应偏导数逼近的误差估计．同时也考虑了整系数Bernstein多项式的L_p模估计．