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关于二元样条空间$S_{3r}^r(\Delta^*)$的维数

高俊斌

(华中理工大学数学系，武汉 430074)

摘　要

设Δ^*是任何三角剖分Δ的HCT细分的三角剖分．本文建立了定义于Δ^*上的二元样条函数空间$S_{3r}^r(\Delta^*)$的维数公式．我们的证明方法同时给出了$S_{3r}^r(\Delta^*)$的一组显示的基函数，并阐明基函数具有某种意义的局部最小支集．

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On the Dimension of the Bivariate Splines Spaces $S_{3r}^r(\Delta^*)$ *

Gao Junbin
(Dept. of Math., Huazhong University of Science and Technology, Wuhan 430074, China)

Abstract We establish the dimensional formula of the space of C^r bivariate piecewise polynomials defined on a triangulation Δ^* which comes from an original triangulation Δ of a connected polygonal domain with HCT subdivision for each triangle of Δ. Our approach is made by constructing a minimal determining set and an associated explicit basis for the space $S_{3r}^r(\Delta^*)$. The minimal determining set is defined well.

Key words multivariate spline, HCT triangulation, B-net, super-spline.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a connected polygonal domain, which does not allowed to contain any hole. Let Δ denote a triangulation of Ω and suppose that the triangles of Δ are labeled by $T^{[1]}, \cdots, T^{[N]}$. For each triangle $T^{[i]} = v_1^{[i]} v_2^{[i]} v_3^{[i]}$ (in counterclockwise order) we take one point $v_0^{[i]}$ in $T^{[i]}$ and connect vertex $v_0^{[i]}$ with vertices $v_k^{[i]}$ $(k = 1, 2, 3)$. It clearly results in a new triangulation Δ^* of Ω called *HCT triangulation* of Δ. In Δ^*, we refer $v_0^{[i]}$ to a new vertex and $v_0^{[i]} v_k^{[i]}$ $(k = 1, 2, 3)$ to new edges of Δ^* for $i = 1, \cdots, N$. Denote $T_k^{[i]} = v_0^{[i]} v_k^{[i]} v_{k+1}^{[i]}$ $(k = 1, 2, 3)$, where $k + 1 \mathrm{mod} 3$. Given $0 \leq r \leq d$, the space of bivariate splines over this triangulation Δ^* is defined by

$$S_d^r(\Delta^*) = \{ s \in C^r(\Omega) : \ s|_{T_k^{[i]}} \in \mathbb{P}_d, \ i = 1, \cdots, N, \ k = 1, 2, 3 \}, \tag{1.1}$$

where $\mathbb{P}_d$ is the $(d+1)(d+2)/2$-dimensional linear space of polynomials of total degree d.

On the dimension of $S_d^r(\Delta^*)$, a lower bound was given by Schumaker (cf. [11]) in terms of the number of interior and boundary vertices of Δ^*. When $d \geq 3r + 2$, C.K. Chui and M.J. Lai (cf. [7]) have proven that such lower bound is exactly the dimension of the space $S_d^r(\Delta^*)$. Since Δ^* is always the triangulation without vertices of degree 4 or 5, we can conclude that the dimension of $S_{3r+1}^r(\Delta^*)$ is equal to the lower bound from the work in my Ph.D Thesis. In this paper we prove the similar results about dimensions of $S_{3r}^r(\Delta^*)$ except $r = 2$. We shall establish an upper bound which agrees with the lower bound and in the process obtain an explicit basis for $S_{3r}^r(\Delta^*)$. Our approach will use Bézier nets to

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construct a certain minimal determining set of domain points. Now we introduce some further notation and present some general results which will be useful for determining the dimension of $S^r_{3r}(\Delta^*)$. Given a triangulation Δ, let Δ^* be a HCT triangulation of Δ, and denote

$$\begin{aligned}
V_I\ (V_I^*) &= \text{number of interior vertices of } \Delta\ (\Delta^*)\\
V_B\ (V_B^*) &= \text{number of boundary vertices of } \Delta\ (\Delta^*)\\
E_I\ (E_I^*) &= \text{number of interior edges of } \Delta\ (\Delta^*)\\
E_B\ (E_B^*) &= \text{number of boundary edges of } \Delta\ (\Delta^*)\\
N\ (N^*) &= \text{number of triangles of } \Delta\ (\Delta^*)\\
E &= E_I + E_B,\quad E^* = E_I^* + E_B^* \ \text{ etc.}
\end{aligned}$$

It is well-known that

$$\begin{aligned}
&E_B = E_B^* = V_B = V_B^*, \quad E_I = 3V_I + V_B - 3\\
&E_I^* = 3V_I^* + V_B^* - 3, \quad N = 2V_I + V_B - 2, \quad N^* = 2V_I^* + V_B^* - 2 \qquad (1.2)\\
&E_I^* = E_I + 3N, \quad V_I^* = V_I + N, \quad N^* = 3N
\end{aligned}$$

We assume that the vertices v_i, $i = 1, \cdots, V^*$ of Δ^* are numbered in such a way that the first V_I of them are original interior vertices, i.e. the interior vertices of Δ, the N of them are interior vertices of Δ^* denoted by $v_{V_I+i} = v_0^{[i]}$, $v_0^{[i]} \in T^{[i]}$, $i = 1, \cdots, N$, and the remaining $V_B^* (= V_B)$ are boundary vertices. For each vertex v_i of Δ^*, let E_i denote the number of edges emanating from v_i, and e_i the number of distinct slopes assumed by these edges. Thus $E_i = e_i = 3$ where $i = V_I + 1, V_I + 2, \cdots, V_I^*$. Then we have the following conclusion.

Lemma 1.1 *Let Δ^* be a HCT triangulation of a given Δ. Then the lower bound for the dimension of $S^r_{3r}(\Delta^*)$ is given by*

$$\begin{aligned}
\dim S^r_{3r}(\Delta^*) \geq \binom{3r+2}{2} + \binom{2r+1}{2} E_I^* - \left[\binom{3r+2}{2} - \binom{r+2}{2}\right] V_I^*\\
+ \sum_{i=1}^{V_I} \sigma_i + N \sum_{j=1}^{r} (r+1-2j)_+ \ := \ lb^*, \qquad (1.3)
\end{aligned}$$

where

$$\sigma_i = \sum_{j=1}^{r} (r+j+1-j\,e_i)_+. \qquad (1.4)$$

2. Preliminaries and tools

Following [3], in order to establish that formula (1.3) provides the actual dimension of $S^r_{3r}(\Delta^*)$, we shall use Bézier net techniques. Associated with any triangulation Δ, let

$$\mathcal{B} = \mathcal{B}_d := \bigcup_{l=1}^{N} \left\{ P_{ijk}^{[l]} = \frac{iv_1^{[l]} + jv_2^{[l]} + kv_3^{[l]}}{d}, \qquad i+j+k=d \right\}, \qquad (2.1)$$

where $v_1^{[l]}$, $v_2^{[l]}$ and $v_3^{[l]}$ are the vertices of the l-th triangle in counterclockwise order (for HCT triangulation Δ^* of Δ, we always take a certain new vertex as $v_1^{[l]}$ in any triangle of Δ^*). The set $\mathcal{B}$ is called the set of Bézier ordinates or domain points. Say that the point $P_{ijk}^{[l]}$'s of distance $d-i$ from the vertex $v_1^{[l]}$ (with similar definitions for the other two vertices). We also say that the point $P_{ijk}^{[l]}$ is of distance i from the edge opposite to $v_1^{[l]}$. The ring of order p around the vertex v is defined as

$$R_p(v) = \{\text{Points which are distance } p \text{ from } v\},$$

and the disk of order p around v is

$$D_p(v) = \bigcup_{j=0}^{p} R_j(v).$$

As all are well-known, each spline $s \in S_d^r(\Delta)$ can be written in the form

$$s(x,y) = s_l(x,y) \qquad \text{for } (x,y) \in T^{[l]}, \quad l = 1, \cdots, N,$$

where each $s_l(x,y)$ is a polynomial of degree d which can be written in Bernstein-Bézier form as follows

$$s_l(\alpha,\beta,\gamma) = \sum_{i+j+k=d} C_{ijk}^{[l]} \frac{d!}{i!\,j!\,k!} \alpha^i \beta^j \gamma^k,$$

where (α,β,γ) are the barycentric coordinates of a point (x,y) in the triangle $T^{[l]}$.

Associated with each domain point $P \in \mathcal{B}$, we define a linear functional on $S_d^r(\Delta)$ by

$$\lambda_P\, s = \text{the coefficient of } s \text{ associated with the domain point } P.$$

The set $\{(P, \lambda_P\, s)\}_{P\in\mathcal{B}}$ is called the Bézier net. If γ is a set of domain points, then we write

$$\Lambda_\gamma = \{\, \lambda_P : \ P \in \Gamma\}.$$

Suppose that $\Gamma \subset \mathcal{B}$ contains m points, and that Λ_Γ has the property that it is a determining set for $S_d^r(\Delta)$ in the sense that

$$s \in S_d^r(\Delta) \text{ and } \lambda\, s = 0 \text{ for all } \lambda \in \Lambda_\Gamma \text{ implies } s = uiv0.$$

Then we have

Lemma 2.1 *dim* $S_{3r}^r(\Delta) \le m$.

The following Lemmas can be found in papers [3] and [7].

Lemma 2.2 *Let v be a boundary vertex of Δ with E edges attached. Suppose that the triangles with vertices at v are numbered countereclockwise as $T^{[1]}, \cdots, T^{[E-1]}$. Finally, let $\mathcal{D}$ denote the following set of domain points:*

1. *All domain points in $T^{[1]} \cap D_p(v)$,*

2. For each $l = 2, \cdots, E-1$, the domain points in the $p-r$ rows of $T^{[l]}$ far away from $T^{[l-1]}$.

Then $\mathcal{D}$ is a minimal determining set for $S_p^r(\Delta)$ on $D_p(v)$ with

$$\#\mathcal{D} = \binom{p+2}{2} + \binom{p-r+1}{2}(E-2).$$

Lemma 2.3 *Let v be an interior vertex of Δ with E edges attached, where e of them have different slopes. Then there exists a subset $\mathcal{D}$ of $D_p(v)$ with*

$$\#\mathcal{D} = \binom{r+2}{2} + \binom{p-r+1}{2}E + \sum_{j=1}^{p-r}(r+j+1-je)_+$$

such that $\mathcal{D}$ determines $S_d^r(\Delta)$ on $D_p(v)$.

The required points in Lemma 2.3 can be given explicitly (cf. [12]). Let $T_1 = v_1v_2v_3$ and $T_2 = v_1v_2v_4$ be shown in Fig.1. Define B-forms

$$P_n = \sum_{i+j+k=n} a_{ijk}\frac{n!}{i!\,j!\,k!}\lambda_1^i\lambda_2^j\lambda_3^k$$

and

$$Q_n = \sum_{i+j+k=n} b_{ijk}\frac{n!}{i!\,j!\,k!}\mu_1^i\mu_2^j\mu_3^k$$

on T_1 and T_2, respectively, where $(\lambda_1, \lambda_2, \lambda_3)$ is the barycentric coordinate of (x, y) with respect to T_1 and (μ_1, μ_2, μ_3) is that with respect to T_2.

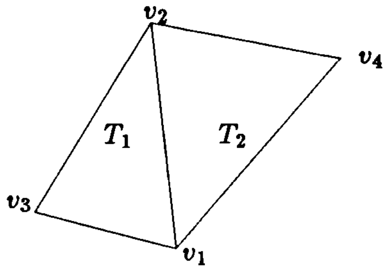

Fig. 1

Lemma 2.4 *Suppose that v_2, v_3 and v_4 are not collinear. For $l \le (n-2)/2$, given Bézier coordinates $\{a_{ijk},\ b_{ijk} : j \ge 1\}$ and $\{a_{ijk} : j = 0,\ 0 \le k \le n-2l-2\}$ which satisfy the smoothness conditions of order up to $n-2l-2$ on v_1v_2, if $\{a_{ijk} : j \ge 1\}$ and $\{b_{ijk} : j \ge 1\}$ satisfy the smoothness conditions of order up to $n-1$ on v_1v_2, then for any given $\{a_{ijk}, b_{ijk} : j = 0, 0 \le i \le l\}$ there exists a unique set of coefficients $\{a_{ijk},\ b_{ijk} : j = 0,\ l+1 \le i \le 2l+1\}$ such that $\{a_{ijk}\}$ and $\{b_{ijk}\}$ meet the C^n smoothness conditions on v_1v_2.*

The proof of Lemma 2.4 is refereed to [7].

If we show that the dimension of $S_{3r}^r(\Delta^*)$ is bounded above by the equality in (1.3), then with Lemma 1.1 the dimension of $S_{3r}^r(\Delta^*)$ will be obtained. The cases where r is odd and even are treated separately in the following two sections.

3. The case for odd r

In this section we always assume that r is odd, say

$$r = 2m + 1$$

By the observations and Lemma 2.1 in Section 2 above, it suffices to construct a determining set for $S_{3r}^{r}(\Delta^{*})$ with the number of elements given in (1.3). First we discuss the case that Δ consists of only one triangle T with three vertices v_1, v_2 and v_3 (in counterclockwise order). Taking one vertex v_0 in T (in general, v_0 is the centroid point of T, i.e. $v_0 = \frac{1}{3}(v_1 + v_2 + v_3)$) and connecting v_0 with v_i $(i = 1, 2, 3)$ result in a triangulation Δ_0 of T. Let $T_l = v_0 v_l v_{l+1}$, where$l = 1, 2, 3$ $(l + 1 \mathrm{mod} 3)$, and denote by $P^{[l]}(x, y)$—the B-form of degree n over T_l such that

$$P^{[l]}(x,y) = \sum_{i+j+k=n} a_{ijk}^{[l]} \frac{n!}{i!\,j!\,k!} \lambda_1^i \lambda_2^j \lambda_3^k, \tag{3.1}$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is the baycentric coordinates of (x, y) with respect to T_l and $a_{ijk}^{[l]}$ associates with the domain point $P_{ijk}^{[l]}$. The following will be devoted to discussing the structure of $S_{3r}^{r}(\Delta_0)$.

Lemma 3.1 *Let $r = 2m + 1$, then*

$$\dim S_{3r}^{r}(\Delta_0) = \binom{2m+3}{2} + 3\binom{4m+3}{2} + 2\binom{m+1}{2} \tag{3.2}$$

This lemma is the deduction of Lemma 2.3. Here we will construct a useful minimal determining set for $S_{3r}^{r}(\Delta_0)$.

Theorem 3.2 *Suppose that $r = 2m + 1$. $\mathcal{P}_0$ denotes the following set of domain points (with respect to $S_{3r}^{r}(\Delta_0)$):*

1. *For each triangle T_l $(l = 1, 2, 3)$, all of the following points of*

$$\left\{ P_{ijk}^{[l]} : i \geq 0,\, k \geq 0,\, j \geq 3m+2 \right\} \bigcup \left\{ P_{ijk}^{[l]} : k \geq 3m+2,\, j \geq 2m+2 \right\}$$
$$\bigcup \left\{ P_{ijk}^{[l]} : i \leq 2m+1,\, j \leq 3m+1,\, k \leq 3m+1 \right\},$$

where $i + j + k = 3r$.

2. *For each triangle T_l $(l = 1, 2, 3)$ all the domain points of*

$$\left\{ P_{ijk}^{[l]} : i \geq 2m+2,\, j \geq m+1,\, k \geq m+1 \right\},$$

where $i + j + k = 3r$.

Then $\mathcal{P}_0$ is a minimal determining set for $S_{3r}^{r}(\Delta_0)$ on Δ_0.

Proof It is not difficult by counting that $\#\mathcal{P}_0 = \dim S_{3r}^{r}(\Delta_0)$. Then it suffices to prove that $s = uiv0$ while $\lambda_P\, s = 0$, for any $P \in \mathcal{P}_0$. First for each vertex v_l $(l = 1, 2, 3)$, it follows from the construction of $\mathcal{P}_0$ and the smoothness conditions of s on $v_0 v_l$ (cf. [8]) that $\lambda_P\, s = 0$, so long as P in $D_{3m+1}(v_l)$. On the other hand, by (1) $\lambda_{P_{ijk}^{[l]}}\, s = 0$, provided that $i \leq 2m + 1$ for $l = 1, 2, 3$. The conditions (2) imply that $\lambda_P\, s$ with P belonging to $\{P^{[l]}_{ijk}: j \le m \text{ or } k \le m\}$ $(l = 1,2,3)$ are undetermined. Moreover, for each vertex v_l $(l = 1,2,3)$, step by step using Lemma 2.4 for the domain points in the $(3m+2)$th row to the $(5m+2)$th row far away from v_l, we can prove that $\lambda_P\, s = 0$ except for P belonging to $\{P^{[l]}_{ijk}: j \le m \text{ and } k \le m\}$. Finally, it follows that $\lambda_P\, s = 0$ for any domain point P of Δ_0 utilizing Lemma 2.4 with regard to v_l and the smoothness conditions of order up to r on v_0v_2 and v_0v_3.

This completes the proof.

Theorem 3.3 *For $S^r_{3r}(\Delta^*)$ $(r = 2m+1)$, let $\mathcal{P}$ consist of the following sets of domain points:*

1. *For each interior vertex v_i $(i = 1,2,\cdots,V_I)$, use Lemma 2.3 to choose a minimal determining set on the disk $D_{3m+1}(v_i)$ for $S^r_{3r}(\Delta^*)$.*
2. *For each boundary vertex v_i $(i = V_I^*+1, V_I^*+2,\cdots,V_I^*+V_B^*)$, use Lemma 2.2 to choose a determining set on $D_{3m+1}(v_i)$.*
3. *For each edge $\eta = v_1v_2$ of Δ, in one of two triangles with η as an edge choose the all domain points which are distance less than $2m+2$ from η but out of $D_{3m+1}(v_1)$ and $D_{3m+1}(v_2)$.*
4. *For each new vertex $v_0^{[l]}$ of Δ^* $(l = 1,2,\cdots,N)$, let $v_0^{[l]}$ be in the triangle $T^{[l]} = v_1^{[l]}v_2^{[l]}v_3^{[l]}$ in counterclockwise order. Choose all of domain points of $D_{4m+1}(v_0^{[l]}) \cap D_{5m+2}(v_k^{[l]}) \cap D_{5m+2}(v_{k+1}^{[l]})$ $(k = 1,2,3)$.*

Then $\mathcal{P}$ is a determining set for $S^r_{3r}(\Delta^*)$ and

$$\begin{aligned}\dim S^r_{3r}(\Delta^*) &= \binom{6m+5}{2} - \left(\binom{6m+5}{2} - \binom{2m+3}{2}\right)(V_I+N)\\ &\quad + \binom{4m+3}{2}(E_I+3N) + 2N\binom{m+1}{2} + \sigma,\end{aligned}$$

where $\sigma = \sum\limits_{i=1}^{V_I}\sigma_i$ and σ_i is defined as (1.4).

Proof We first check the cardinality of $\mathcal{P}$. The cardinality of sets in (1), (2), (3) and (4) are, respectively,

$$\sum_{i=1}^{V_I}\left[\binom{2m+3}{2} + \binom{m+1}{2}E_i + \sigma_i\right],$$
$$\sum_{j=1}^{V_B}\left[\binom{3m+3}{2} + \binom{m+1}{2}(E_{V_I^*+j} - 2)\right],$$
$$\sum_{i=1}^{E}\binom{2m+2}{2} \quad \text{and} \quad 3N\binom{2m+1}{2}.$$

Thus we see that the cardinality of $\mathcal{P}$ is given by

$$\#\mathcal{P} = \left(\binom{3m+3}{2} - 2\binom{m+1}{2}\right) V_B + (2E+3N)\binom{m+1}{2}$$
$$\binom{2m+3}{2} V_I + 3N\binom{2m+1}{2} + \binom{2m+2}{2} E + \sum_{i=1}^{V_I} \sigma_i$$

Using (2.1), it is easy to see that this formula reduces to the expression in (1.3).

It remains to check that $\mathcal{P}$ is a determining set for $S_{3r}^r(\Delta^*)$. For each vertex v_i ($i = 1, \cdots, V_I, V_I^* + 1, \cdots, V_I^* + V_B^*$), it is clear that $\lambda_P\, s = 0$ for any $P \in D_{3m+1}(v_i)$ on Δ^*, because of the points of $\mathcal{P}$ chosen in items (1) and (2). For each triangle of Δ^*, some one edge $\eta = v_1 v_2$ ($v_1, v_2 \in \Delta$) of that triangle must be the edge of Δ. On this triangle the set $\mathcal{A}$ of all domain points which are of distance less than $2m+2$ from η but out of $D_{3m+1}(v_1)$ or $D_{3m+1}(v_2)$, is contained in $\mathcal{P}$ or not. Thus we can see that $\lambda_P\, s = 0$ for $P \in \mathcal{A}$ with (3) or the smoothness conditions of order up to r on η. Finally, by Theorem 3.2, we may prove that $\lambda_P\, s = 0$ for any domain point P in each triangle of Δ. Thus $s = uiv0$, because all of the Bézier coordinate of s are zero and the theorem is established.

The following theorem follows immediately from Theorem 3.3.

Theorem 3.4 *There exists a local explicit basis of $S_{3r}^r(\Delta^*)$, say $A(\mathcal{P}) = \{ \mathcal{B}_p \in S_{3r}^r(\Delta^*) : \lambda_Q \mathcal{B}_P = \Delta_{QP}$, for any $P, Q \in \mathcal{P}\}$. And when P is in the set of items (1) and (2) of Theorem 3.3,* supp$\mathcal{B}_P$ *consists of all of the triangles of Δ with vertices at v_i. For P in the set of item (3) of Theorem 3.3,* supp$\mathcal{B}_P$ *consists of two triangles of Δ with η as an edge, and for P in the set of item (4)* supp$\mathcal{B}_P$ *is only one triangle of Δ which contains P.*

4. The case for even r

In this section we will establish the similar conclusions as the preceding section in the case where r is even, say

$$r = 2m \qquad (m > 1)$$

Let Δ_0, etc. be defined as the preceding section. Then from Lemma 2.2, it follows that

Lemma 4.1 *Suppose that $r = 2m$, then*

$$\dim S_{3r}^r(\Delta_0) = \binom{2m+2}{2} + 3\binom{4m+1}{2} + m^2. \tag{4.1}$$

Now we give a useful minimal determining set for $S_{3r}^r(\Delta_0)$.

Theorem 4.2 *Let $\mathcal{P}_0$ denote the following set of domain points:*

1. *For each triangle T_l ($l = 1,2,3$), choose all of the domain points of*

$$\left\{ P_{ijk}^{[l]} : j \ge 3m+1 \right\} \bigcup \left\{ P_{ijk}^{[l]} : j \ge 3m+1,\ k \ge 3m+1 \right\}$$
$$\bigcup \left\{ P_{ijk}^{[l]} : i \le 2m,\ j \le 3m,\ k \le 3m \right\}$$

except for $P_{(2m,3m,m)}^{[l]}$, where $i+j+k = 3r$.

2. For each triangle T_l $(l=1,2,3)$, choose all of the points of

$$\left\{ P^{[l]}_{ijk} : i \geq 2m+1,\, j \geq m+1,\, k \geq m+1 \right\} \bigcup \left\{ P^{[l]}_{ijk} : m+1 \leq j \leq 3m-1,\, i+j=6m \right\},$$

where $i+j+k=3r$.

3. For only one triangle, say T_1, choose $P^{[1]}_{(5m,m,0)}$.

Then $\mathcal{P}_0$ is a determining set for $S^r_{3r}(\Delta_0)$ on Δ_0.

Proof It is not difficult by counting that $\#\mathcal{P}_0 = \dim S^r_{3r}(\Delta_0)$. Then it suffices to prove that $s = uiv0$ while $\lambda_P = 0$, for any $P \in \mathcal{P}_0$. First for each vertex v_l $(l=1,2,3)$, on $R_{3m}(v_l)$, the Bézier ordinates associated with the domain points in $\{ P^{[l]}_{ijk} : j=3m, 0 \leq k \leq m \} \bigcup \{ P^{[l]}_{ijk} : 1 \leq j \leq m-1, k=3m \}$ are undetermined. The number of those ordinates is $2m$. Using C^{2m} smoothness conditions on $R_{3m}(v_l)$ results in that all of ordinates are zero. Second, along $R_j(v_l)$ $(3m+1 \leq j \leq 5m)$, according to C^{2m} smoothness conditions (cf. Lemma 2.4), we can conclude that all Bézier ordinates except for that associated with the doamin points in $\{ P^{[l]}_{ijk} : 0 \leq j \leq m, 0 \leq k \leq m \}$, on each triangle T_l, are zero. Finally by, using C^{2m} smoothness conditions on $R_{5m+1}(v_1)$ and then on v_0v_l $(l=1,2,3)$ we will show that give all Bézier ordinates are zero. This completes the proof.

Given a triangulation Δ_{2n}, in which there is only one interior vertex v, be shown in Fig.2. Let θ_i =angle v_ivv_{i+1} $(i \in \mathbb{Z}_n)$ in counterclockwise order. If $n \geq 4$, then there exists $i_0 \in \mathbb{Z}_n$ such that $\theta_{i_0} + \theta_{i_0+1} < \pi$. Without loss of the generality, suppose that $i_0 = 1$.

Let $v_2 = \alpha_1 v + \beta_1 v_3 + \gamma_1 w_2$, $v_2 = \alpha_2 v + \beta_2 v_1 + \gamma_2 w_1$, and $w_2 = \alpha_0 v + \beta_0 w_1 + \gamma_0 v_2$, then we have the conditions

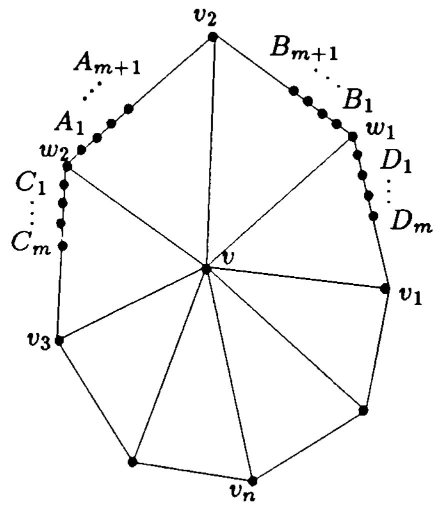

Fig. 2

$$\gamma_0\gamma_1\gamma_2 + \gamma_1\beta_0 - \gamma_2 \geq 0 \tag{4.2}$$

and

$$\gamma_i > 0 \tag{4.3}$$

Let $n \geq 4$ and the above conditions are satisfied. We define the following domain point set $\mathcal{D}$ on $D_{3m}(\Delta_{2n})$:

1. The minimal determining set on $D_{3m-1}(\Delta_{2n})$ for $S^{2m}_{3m-1}(\Delta_{2n})$;
2. On boundary edges of Δ_{2n}, in counterclockwise order, take $2m+1$ points from v_1, $2m-1$ points from v_2, m points from v_i and w_i where $i \geq 3$, respectively.

Lemma 4.3 *The $\mathcal{D}$ is a minimal determining set for $S^{2m}_{3m}(\Delta_{2n})$ on Δ_{2n}.*

Proof On Δ_{2n} the remaining undetermined Bézier ordinates are $A_1, \cdots, A_m, A_{m+1}$, $B_1, \cdots, B_m, B_{m+1}$, $C_1, \cdots, C_m$ and $D_1, \cdots, D_m$ by construction of $\mathcal{D}$ and smoothness conditions. We will see that smoothness conditions on vv_2, vw_2 and vw_1 result in the following system:

$$\begin{aligned}
&A_{m+1} = \beta_0^{2m-1} B_{m+1}\\
&A_m = \beta_0^{2m} B_m + 2m\beta_0^{2m-1}\gamma_0 B_{m+1}\\
&\binom{2m}{0}\gamma_1^{2m}C_1 + \binom{2m}{1}\gamma_1^{2m-1}\beta_1 C_2 + \cdots + \binom{2m}{m-1}\gamma_1^{m+1}\beta_1^{m-1}C_m = 0\\
&\cdots \qquad \cdots \qquad \cdots\\
&\binom{m+2}{0}\gamma_1^{m+2}C_1 + \binom{m+2}{1}\gamma_1^{m+1}\beta_1 C_2 + \cdots + \binom{m+2}{m-1}\gamma_1^{3}\beta_1^{m-1}C_m = 0\\
&\binom{m+1}{0}\gamma_1^{m+1}C_1 + \binom{m+1}{1}\gamma_1^{m}\beta_1 C_2 + \cdots + \binom{m+1}{m-1}\gamma_1^{2}\beta_1^{m-1}C_m = A_{m+1}\\
&\binom{m}{0}\gamma_1^{m}C_1 + \binom{m}{1}\gamma_1^{m-1}\beta_1 C_2 + \cdots + \binom{m}{m-1}\gamma_1^{1}\beta_1^{m-1}C_m = A_m\\
&\binom{2m}{0}\gamma_2^{2m}D_1 + \binom{2m}{1}\gamma_2^{2m-1}\beta_2 D_2 + \cdots + \binom{2m}{m-1}\gamma_2^{m+1}\beta_2^{m-1}D_m = 0\\
&\cdots \qquad \cdots \qquad \cdots\\
&\binom{m+2}{0}\gamma_2^{m+2}D_1 + \binom{m+2}{1}\gamma_2^{m+1}\beta_2 D_2 + \cdots + \binom{m+2}{m-1}\gamma_2^{3}\beta_2^{m-1}D_m = 0\\
&\binom{m+1}{0}\gamma_2^{m+1}D_1 + \binom{m+1}{1}\gamma_2^{m}\beta_2 D_2 + \cdots + \binom{m+1}{m-1}\gamma_2^{2}\beta_2^{m-1}D_m = B_{m+1}\\
&\binom{m}{0}\gamma_2^{m}D_1 + \binom{m}{1}\gamma_2^{m-1}\beta_2 D_2 + \cdots + \binom{m}{m-1}\gamma_2^{1}\beta_2^{m-1}D_m = B_m.
\end{aligned}$$

It follows that

$$\begin{aligned}
\gamma_1 A_m &= m A_{m+1}, & \gamma_2 B_m &= m B_{m+1}\\
A_{m+1} &= \beta_0^{2m-1} B_{m+1}, & A_m &= \beta_0^{2m} B_m + 2m\beta_0^{2m-1}\gamma_0 B_{m+1}.
\end{aligned}$$

According to the conditions (4.2) and (4.3) we conclude that $A_m = A_{m+1} = B_m = B_{m+1} = 0$, then all A's, B's, C's, and D's are zero. This completes the proof.

Lemma 4.4 *For $n = 3$, we take the minimal determining set for $S^{2m}_{3m}(\Delta_6)$ defined as in [12].*

We first note the fact that if v is an interior vertex of $\deg(v) = 3$ in Δ and vw is an edge of Δ (where w is another interior vertex of Δ), then $\deg(w) \geq 4$. Now we are in a position to state the main results.

Theorem 4.5 *For $S^r_{3r}(\Delta^*)$ $(r = 2m)$, let $\mathcal{P}$ be the union of the following sets of domain points:*

1. *For each interior vertex v_i with $deg(v_i) = 3$ in Δ, use Lemma 4.4 to choose a minimal determining set for $S^r_{3r}(\Delta^*)$ on the disk $D_{3m}(v_i)$ and mark the edge of Δ whose middle point is not chosen in the process;*
2. *For each other interior vertex v_i of Δ, use Lemma 4.3 to choose a determining set for $S^r_{3r}(\Delta^*)$ on $D_{3m}(v_i)$ such that v_1 in Lemma 4.3 corresponds to a certain one vertex in Δ, except for the middle points of edges marked;*
3. *For each boundary vertex v_i $(i = V_I + 1, \cdots, V_I + V_B)$, use Lemma 2.2 to choose a minimal determining set on $D_{3m}(v_i)$;*
4. *For each edge $\eta = v_1v_2$ of Δ, in one of two triangles with η as an edge choose all domain points which are distance less than $2m+1$ from η but out of $D_{3m}(v_1)$ and $D_{3m}(v_2)$;*
5. *For each triangle of Δ, use items (2) and (3) of Theorem 4.2 to choose all of corresponding domain points.*

Then $\mathcal{P}$ is a minimal determining set for $S^r_{3r}(\Delta^*)$.

Proof Clearly we can see that the cardinlity of $\mathcal{P}$ is given by

$$\begin{aligned}
\#\mathcal{P} &= \left(\binom{3m+2}{2} - 2\binom{m+1}{2}\right)V_B + \sum_{i=1}^{V_I}\sigma_i + (2E+3N)\binom{m+1}{2} \\
&\quad \binom{2m+2}{2}V_I + E\binom{2m}{2} + 3N\left(\binom{2m-1}{2} + (2m-1)\right) + N - E \\
&= \binom{6m+2}{2} + \binom{4m+1}{2}E_I^* + \left(\binom{2m+2}{2} - \binom{6m+2}{2}\right)V_I^* \\
&\quad +Nm^2 + \sum_{i=1}^{V_I}\sigma_i.
\end{aligned}$$

This quality is equal to that of (1.3) for $r = 2m$. The proof for the determining property of $\mathcal{P}$ is very similar to that in the odd case, so we omit it here.

Theorem 4.6 *For $r = 2m$ $(m > 1)$ there exists a local basis of space $S^r_{3r}(\Delta^*)$, say*

$$A(\mathcal{P}) = \{\mathcal{B}_P \in S^r_{3r}(\Delta^*) : \ \lambda_Q\,\mathcal{B}_P = \Delta_{PQ},\ P, Q \in \mathcal{P}\}$$

And when P is in the set of items (1), (2) and (3) of Theorem 4.4, supp$\mathcal{B}_P$ *consists of all triangles with a vertex at v_i or another vertex u_i adjacent to v_i such that $P \in D_{3m}(u_i)$.*

For P in the set of item (4) suppβ_P *composes of two triangles of* Δ *with* η *as an edge and for P pn the set of item (5)* suppβ_P *is only one triangle of* Δ *containing* P.

5. Super-spline subspaces

Super-spline subspaces were introduced and studied by some authors. In order to discuss some subspaces of $S^r_{3r}(\Delta^*)$, following [3], we can extend the concept of super-spline.

Definition 5.1 *Let* Δ^* *be the HCT triangulation of a given triangulation* Δ. *We define the subspace of* $S^r_d(\Delta^*)$ *with enhanced smoothness* $\rho \geq r$ *at the vertices of* Δ *by*

$$S^{r,\rho}_d(\Delta^*) = \{s \in S^r_d(\Delta^*): \quad s \in C^\rho(v_i), i = 1;\cdots, V_I\}, \tag{5.1}$$

where $C^\rho(v) = \{s: \ s$ *has derivatives up to order* ρ *at* $v\}$. *We refer* $s \in S^{r,\rho}_d(des)$ *to quasi-super-spline of* $S^r_d(\Delta^*)$.

Theorem 5.2 *Let* Δ^* *be the HCT triangulation of* Δ. *Then we have the following dimensinal formulae:*

$$\dim S^{2m,3m-1}_{6m}(\Delta^*) = \binom{3m+1}{2}V + \left(\binom{2m}{2} + (2m-1)\right)E \tag{1}$$

$$3\left(\binom{2m-1}{2} + (3m-1)\right)N + N + V_s + (m+1)V_B \tag{2}$$

and

$$\dim S^{2m+1,3m+1}_{6m+3}(\Delta^*) = \binom{3m+3}{2}V + \binom{2m+2}{2}E + 3\binom{2m+1}{2}N.$$

Remarks:

1. It is also possible to give similar results for the case where the enhanced smoothness order $r \leq \rho \leq 3m-1$ for $r = 2m$ and $r \leq \rho \leq 3m+1$ for $r = 2m+1$ as in [3].

2. In [7] the interpolation problems for super-spline subspaces of $S^r_d(\Delta)$ with $d \geq 3r+2$ were discussed. We may study some interpolation schemes for $S^{2m,3m-1}_{6m}(\Delta^*)$ and $S^{2m+1,3m+1}_{6m+3}(\Delta^*)$ whose interpolants can be calculated out explicitly, and also can proceed the approximation theorem by subspaces $S^{2m,3m-1}_{6m}(\Delta^*)$ and $S^{2m+1,3m+1}_{6m+3}(\Delta^*)$.

3. When $r = 2$, the dimension formula about the spline space $S^2_6(\Delta^*)$ may be established necessarily in other ways. In [9], we construct an explicit interpolation scheme for $S^2_6(\Delta^*)$, but the scheme is not local. We conjecture that in general there doesn't exist a local basis for $S^2_6(\Delta^*)$.

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关于二元样条空间$S_{3r}^r(\Delta^*)$的维数

高俊斌

(华中理工大学数学系，武汉 430074)

摘 要

设Δ^*是任何三角剖分Δ的HCT细分的三角剖分. 本文建立了定义于Δ^*上的二元样条函数空间$S_{3r}^r(\Delta^*)$的维数公式. 我们的证明方法同时给出了$S_{3r}^r(\Delta^*)$的一组显示的基函数，并阐明基函数具有某种意义的局部最小支集.