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Kornai-Weibull 排队模型的纰漏，修正与分析

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摘 要

本文首先指出Kornai-Weibull 排队模型(A) 中之接待率$r(y_1)$ 的定义是不正确的，并给出接待率$r(y_1,y_2)$ 的正确定义. 其次，证明了修正模型(1) 有“正常状态”的充要条件是接待能力

$$S < S^* \triangleq \frac{\lambda\gamma\chi n}{\lambda\chi + (1-\lambda)\gamma}.$$

于是，本文发现接待能力S 的分歧值是S^*; 当$S < S^*$ 时，市场是短缺的; 当$S \geq S^*$ 时，市场是不短缺的.

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Careless Mistake, Correction and Analysis of the Kornai-Weibull Queue Model *

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Abstrict We find that the definition of the service rate $r(y_1)$ in Kornai-Weibull's queueing model (A) is incorrect, and the correct definition of the service rate $r(y_1, y_2)$ is given. We prove that the revised model (1) have a 'mormal state' if and only if service capacity $S < \lambda\gamma\chi n/(\lambda\chi + (1-\lambda)\gamma)$.

Key words Kornai-Weibull queue model, correction, analysis.

I. Queueing model

Kornai-Weibull's queueing model of a market in a shortage economy is a part of Kornai's theory of shortage, one of the most important economic therories developed recently. Kornai and Weibull (1977) establish and research the model, which is described by a system of ordinary differential equations [also see Kornai and weibull (1978). Kornai (1980)]. Global stability of a stationary point of this system has a very important interpretation in the theory of shortag, where the stationary point describes a normal state of the market.

Only in view of a simple model. Kornai and Weibull try to give the existence proof of normal state by strict logical reasoning. Here, let us redescribe this model. We consider a market trading one good only. The price of the good is constant. This good is being traded in indivisible items. Each buyer can buy only one item. There is only one seller and there are n buyers. There is substitute to be sold in another market. It is assumed that the substitute is cheaper than the good and is available without queueing. First of all, a buyer must decide whether the price of a good is acceptable for him in the process of shopping. If he doesn't accept the price, he buys a substitute. If he accepts the price, he considers the queueing time. If he accepts the queueing time, he joins the queue. Otherwise, he buys the substitute. So, at any time, every buyer is either queueing, or consuming the good, or consuming the substitute.

At any time t, we denote by $y_1(t)$ the number of queueing buyers, by $y_2(t)$ the number of buyers consuming the good, by $y_3(t)$ the number of buyers consuming the substitute.

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The process of shoping is described by the following system of equations.

$$\begin{cases} \frac{dy_1}{dt} = \lambda\varphi(y_1)(\gamma y_2 + \chi y_3) - r(y_1), \\ \frac{dy_2}{dt} = r(y_1) - \gamma y_2, \\ \frac{dy_3}{dt} = (1 - \lambda\varphi(y_1))(\gamma y_2 + \chi y_3) - \chi y_3, \end{cases} \tag{A}$$

where λ is the initial buying propensity and depends only on the price of the good, $\varphi(y_1)$ is the queueing propensity and depends on the queue length y_1, γ is the need-renewal rate of the good, and χ is the need-renewal rate of the substitute ($1/\gamma$ is average good satisfaction time, and $1/\chi$ is average substitute satisfaction time), $r(y_1)$ is the service rate, the number of buyers served per time unit. Kornai and Weibull consider that it depends only on the queue length y_1, and denote

$$r(y_1) = \begin{cases} S & \text{if} \quad y_1 > 0, \\ 0 & \text{if} \quad y_1 = 0, \end{cases}$$

where S is constant. It is called the service capacity, the maximal number of buyers served per time unit. $M = (n, S, \lambda, \varphi, \gamma, \chi)$ is defined as a market. A three-dimensional vector $y = (y_1, y_2, y_3)$ is called state of the market. A set $Y = \{y : y \in R^3,\ y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ y_1 + y_2 + y_3 = n\}$ is called a set of feasible states of the market M. We know that variables y_1, y_2 and y_3 is dependent from $y_1 + y_2 + y_3 = n$. Relevantly, the sum of three equations of system (A) is $\frac{d}{dt}(y_1 + y_2 + y_3) = 0$. Putting $y_3 = n - y_1 - y_2$, we have a system

$$\frac{dy}{dt} = f(y), \tag{B}$$

where $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$,

$$f(y) = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix} \triangleq \begin{pmatrix} \lambda\varphi(y_1)[\gamma y_2 + \chi(n - y_1 - y_2)] - r(y_1) \\ r(y_1) - \gamma y_2 \end{pmatrix} \tag{C}$$

and a set of reduced feasible states $\bar{Y} = \{y : y \in R^2,\ y_1 \geq 0,\ y_2 \geq 0,\ y_1 + y_2 \leq n\}$. Here, the question is: Under what conditions, system (B) has the following four properties?

(1) There exists exactly one stationary point $\bar{y}$ of system (B), and $\bar{y} \in \text{Int}\ \bar{Y}$;

(2) The stationary point $\bar{y}$ of system (B) is locally asymptotically stable;

(3) For every $y_0 \in \bar{Y}$, there exists exactly one solution $y(t; y_0)$ of system (B) such that $y|_{t=0} = y_0$ and $y(t; y_0)$ is defined on $[0, \infty)$ and $y(t) \in \bar{Y}$ for all $t \geq 0$;

(4) for every $y_0 \in \bar{Y}$, $\lim_{t \to +\infty} y(t; y_0) = \bar{y}$.

by assumption (IV) we obtain that $-1 \geq -\hat{y}_1 \frac{1-\lambda\chi}{\lambda\chi n}$ and $\hat{y}_1 \geq \frac{\lambda\chi n}{1-\lambda\chi}$. Noting that

$$y_1^* \leq \frac{S^* - \lambda\gamma n\varphi(y_1^*)}{1-\lambda\chi} < \lambda\chi n \frac{1-\frac{\gamma}{\chi}\varphi(y_1^*)}{1-\lambda\chi} \leq \frac{\lambda\chi n}{1-\lambda\chi} \leq \hat{y}_1,$$

we have that for $0 \leq y_1 \leq y^*$, $\varphi(y_1) > 0$. Therefore, if $y \in \bar{Y}$, $0 \leq y_1 \leq y_1^*$, then $\frac{\partial h(y_1,y_2)}{\partial y_2} = -\lambda(\chi-\gamma)\varphi(y_1) < 0$. Since $\frac{\partial h(y_1,0)}{\partial y_1} = 1 + \lambda\chi\varphi'(y_1)(n-y_1) - \lambda\chi\varphi(y_1)$ $\geq \frac{1-\lambda\chi}{n} y_1$, then

$$h(y_1,0) \geq h(0,0) = \lambda\chi n > S^* > S \quad (0 \leq y_1 \leq n).$$

For every $y_1 \in [0, y_1^*]$, there exists only one $y_2 = \frac{y_1 + \lambda\chi\varphi(y_1)(n-y_1) - S}{\lambda(\chi-\gamma)\varphi(y_1)} \triangleq \psi_1(y_1)$ such that $h(y_1,y_2) = S$ and $0 < y_2 = \psi_1(y_1) < n - y_1$ $(0 \leq y_1 < y_1^*)$. Let S_1 denotes the curve fulfilled $y_2 = \psi_1(y_1)$ $(0 \leq y_1 \leq y_1^*)$. Because, for $0 \leq y_1 \leq y_1^*$, $\frac{\partial h(y_1,y_2)}{\partial y_2} < 0$; for $y_1^* < y_1 \leq n$, $\frac{h(y_1,y_2)}{\partial y_2} \leq 0$, then $D_1 = \{y : 0 \leq y_1 \in< y_1^*,\ \psi_1(y_1) < y_2 \leq n - y_1\}$, $D_2 = \{y : 0 \leq y_2 \leq \psi_1(y_1)\ (0 \leq y_1 < y_1^*),\ 0 \leq y_2 \leq n - y_1\ (y_1^* \leq y_1 \leq n)\}$. By the above demonstration, we obtain that

Lemma 2 *If $\lambda < 1, \lambda\gamma n < S < S^*$, then*

$$r(y_1,y_2) = \begin{cases} y_1 + \lambda\varphi(y_1)[\gamma y_2 + \chi(n - y_1 - y_2)], & \text{for } \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in D_1, \\ S & \text{for } \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in D_2. \end{cases}$$

Lemma 3 *The curve $S_1 : y_2 = \psi_1(y_1)$ $(0 \leq y_1 \leq y_1^*)$ is above the line $S_2 : y_2 = S/\gamma$ $(0 \leq y_1 \leq n - \frac{S}{\gamma})$. That is to say, $\psi_1(y_1) > S/\gamma$ $(0 \leq y_1 \leq y_1^*)$.*

Proof First, $\psi_1(0) = \frac{\lambda\chi n - S}{\lambda(\chi-\gamma)} \triangleq y_2^*$. Obviously, $y_2^* > \frac{\lambda\chi n - S^*}{\lambda(\chi-\gamma)} = \frac{\lambda\chi n}{\lambda\chi + (1-\lambda)\gamma} = \frac{S^*}{\gamma}$ $> \frac{S}{\gamma}$. So, if we demonstrate $\frac{d\psi_1(y_1)}{dy_1} \geq 0$ $(0 \leq y_1 \leq y_1^*)$ the Lemma is proved. In fact, if $0 \leq y_1 \leq y_1^*$, then

$$\begin{aligned} \frac{d\psi_1(y_1)}{dy_1} &= \frac{1}{\lambda(\chi-\gamma)\varphi^2(y_1)}\{[1 + \lambda\chi\varphi'(y_1)(n-y_1) - \lambda\chi\varphi(y_1)]\varphi(y_1) \\ &\quad - [y_1 + \lambda\chi\varphi(y_1)(n-y_1) - S]\varphi'(y_1)\} \\ &= \frac{1}{\lambda(\chi-\gamma)\varphi^2(y_1)}[(1-\lambda\chi\varphi(y_1))\varphi(y_1) + (S-y_1)\varphi'(y_1)]. \end{aligned}$$

From assumption (IV), we obtain

$$\varphi(y_1) = \int_0^{y_1} \varphi'(\tau)d\tau + 1 \geq 1 - \frac{1-\lambda\chi}{\lambda\chi n} y_1.$$

According to the service rate $r(y_1)$ difined by Kornai and Weibull, none good can be sold in the market without queueing, and the number of buyers consuming goods is monotonic decreasing. Obviously, it is very absurd.

The revised queueing model is

$$\begin{cases} \dfrac{dy_1}{dt} = \lambda\varphi(y_1)[\gamma y_2 + \chi y_3] - r(y_1, y_2), \\ \dfrac{dy_2}{dt} = r(y_1, y_2) - \gamma y_2, \\ \dfrac{dy_3}{dt} = (1 - \lambda\varphi(y_1))(\gamma y_2 + \chi y_3) - \chi y_3, \end{cases} \tag{1}$$

and

$$\frac{dy}{dt} = f(y), \tag{2}$$

where $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$,

$$f(y) = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix} \triangleq \begin{pmatrix} \lambda\varphi(y_1)[\gamma y_2 + \chi(n - y_1 - y_2)] - r(y_1, y_2) \\ r(y_1, y_2) - \gamma y_2 \end{pmatrix}. \tag{3}$$

We shall assume that

(I) λ, γ, χ and S are constants, and $0 < \lambda \leq 1,\ 0 < \gamma < \chi,\ 0 < S$;

(II) The function $\varphi : [0, \infty) \to [0, 1]$ is nonincreasing of class C^1, with $\varphi(0) = 1$, $\varphi(y_1) < 1\ \ (y_1 > 0)$, $\varphi(y_1) = 0\ \ (y_1 \geq n)$;

(III) $0 < \gamma < \chi < 1/2$;

(IV) $0 \geq \varphi'(y_1) \geq -\dfrac{1 - \lambda\chi}{\lambda\chi n}$.

Assumptions (I), (II) are original and practical. Assumption (III) is easy to bring about so long as we select suitable unit of time t. Assumption (IV) imposes restrictions on the speed at which the queueing propensity drops with the increase of the queue length y_1. It is reasonable also.

The main result of this paper is the following theorem.

Theorem *Let assumptions (I),(II),(III) and (IV) hold, if* $S < S^* \triangleq \dfrac{\lambda\chi\gamma n}{\lambda\chi + (1 - \lambda)\gamma}$, *then system (2) has only unique stationary point* $\bar{y} \in$ *Int* $\bar{Y}$ *and is globally asymptotically stable in the set* $\bar{Y}$. *On the contrary, if system (2) has only unique stationary point* $\bar{y} \in$ *Int* $\bar{Y}$ *on* $\bar{Y}$, *then we have* $S < S^*$.

According to the theorem, we know that, if assumptions (I),(II),(III) and (IV) are fulfilled, $S < S^*$ is a necessary and sufficient condition under which system (1) has the normal state. From $\bar{y} \in$ Int $\bar{Y}$ we obtain that, $\bar{y_1} > 0$, where $\bar{y_1}$ is the average queue length. $\bar{t} = \bar{y_1}/s$ we call the average queueing time. For $\bar{y_1} > 0, \bar{t} > 0$. $\bar{t}$ mirrors exactly a basic feature of a market in a shortage economy. Then S^* is a quantity index. By it, we can judge whether a market is short or not. That is to say, if $S < S^*$, the market is short, and if $S \geq S^*$ the market is not short. In words, S^* is a branch value by which we can judge whether a market is short or not.

2. Proof of theorem

At first, we analyse the valuation of the function $r(y_1, y_2)$ in system (2) in $\bar{Y}$. In $\bar{Y}$, we obtain from $0 < \gamma < \chi$ that

$$\frac{\partial h(y_1, y_2)}{\partial y_2} \leq 0.$$

If $0 \leq y_1 \leq n$, then

$$h(y_1, y_2) \geq h(y_1, n - y_1) = y_1 + \lambda\varphi(y_1)\gamma(n - y_1) \triangleq \psi(y_1).$$

By assumptions (II) and (IV), We obtain that

$$\frac{d\psi(y_1)}{dy_1} = 1 - \lambda\varphi(y_1)\gamma + \lambda\gamma(n - y_1)\varphi'(y_1) \geq 1 - \lambda\gamma + \lambda\gamma n\varphi'(y_1) \geq 1 - \frac{\gamma}{\chi} > 0, \quad (0 \leq y_1 \leq n).$$

For $y \in \bar{Y}$, we always have

$$h(y_1, y_2) \geq \psi(y_1) \geq \psi(0) = h(0, n) = \lambda\gamma n.$$

Actually, we have proved the following lemma.

Lemma 1 *If $\lambda = 1, S < \gamma n$ or $\lambda < 1$, $S \leq \lambda\gamma n$ we always have $r(y_1, y_2) = S$ in $\bar{Y}$.*

If $\lambda < 1$, $\gamma\lambda n < S < S^* \triangleq \dfrac{\lambda\chi\gamma n}{\lambda\chi + (1 - \lambda)\gamma}$, the valuation of $r(y_1, y_2)$ in $\bar{Y}$ becomes more complicated. Obviously,

$$\partial\bar{Y} = L_1 \cup L_2 \cup L_3,$$

where

$$\begin{aligned} L_1: &\quad y_1 = 0 &\quad (0 \leq y_2 \leq n),\\ L_2: &\quad y_2 = 0 &\quad (0 \leq y_1 \leq n),\\ L_3: &\quad y_2 = n - y_1 &\quad (0 \leq y_1 \leq n). \end{aligned}$$

From $h(0, n) = \lambda\gamma n < S$, $h(n, 0) = n > \lambda\chi n > S^* > S$, we obtain that both the set $D_1 = \{y : y \in \bar{Y},\ h(y_1, y_2) < S\}$ and the set $D_2 = \{y : y \in \bar{Y},\ h(y_1, y_2) \geq S\}$ are non-empty, and $\bar{Y} = D_1 \cup D_2$.

On L_3, $h(y_1, n - y_1) = y_1 + \lambda\gamma\varphi(y_1)(n - y_1) = \psi(y_1)$. Since $\dfrac{d\psi(y_1)}{dy_1} > 0 \ \ (0 \leq y_1 \leq 0)$, $\psi(0) = \lambda\gamma n < S$, $\psi(n) = n > S$, there exists exactly one point $y_1^* \in (0, n)$ such that $\psi(y_1^*) = h(y_1^*, n - y_1^*) = S$, and $y_1^* = \dfrac{S - \lambda\gamma n\varphi(y_1^*)}{1 - \lambda\gamma\varphi(y_1^*)}$. So, for $0 \leq y_1 < y_1^*$, $h(y_1, n - y_1) < S$; for $y_1^* < y_1 \leq n$, $h(y_1, n - y_1) > S$. From assumption (II) we know that, there exists $\hat{y}_1$ such that $\varphi(y_1) > 0 \ \ (0 \leq y_1 < \hat{y}_1)$, $\varphi(y_1) = 0 \ \ (\hat{y}_1 \leq y_1)$, $\hat{y}_1 \in (0, n]$.

From the equality

$$0 = \varphi(\hat{y}_1) = \int_0^{\hat{y}_1} \varphi'(y_1)dy_1 + 1,$$

by assumption (IV) we obtain that $-1 \geq -\hat{y}_1 \dfrac{1-\lambda\chi}{\lambda\chi n}$ and $\hat{y}_1 \geq \dfrac{\lambda\chi n}{1-\lambda\chi}$. Noting that

$$y_1^* \leq \frac{S^* - \lambda\gamma n\varphi(y_1^*)}{1-\lambda\chi} < \lambda\chi n \frac{1-\frac{\gamma}{\chi}\varphi(y_1^*)}{1-\lambda\chi} \leq \frac{\lambda\chi n}{1-\lambda\chi} \leq \hat{y}_1,$$

we have that for $0 \leq y_1 \leq y^*$, $\varphi(y_1) > 0$. Therefore, if $y \in \bar{Y}$, $0 \leq y_1 \leq y_1^*$, then $\dfrac{\partial h(y_1,y_2)}{\partial y_2} = -\lambda(\chi-\gamma)\varphi(y_1) < 0$. Since $\dfrac{\partial h(y_1,0)}{\partial y_1} = 1 + \lambda\chi\varphi'(y_1)(n-y_1) - \lambda\chi\varphi(y_1)$ $\geq \dfrac{1-\lambda\chi}{n} y_1$, then

$$h(y_1,0) \geq h(0,0) = \lambda\chi n > S^* > S \quad (0 \leq y_1 \leq n).$$

For every $y_1 \in [0, y_1^*]$, there exists only one $y_2 = \dfrac{y_1 + \lambda\chi\varphi(y_1)(n-y_1) - S}{\lambda(\chi-\gamma)\varphi(y_1)} \triangleq \psi_1(y_1)$ such that $h(y_1,y_2) = S$ and $0 < y_2 = \psi_1(y_1) < n - y_1$ $(0 \leq y_1 < y_1^*)$. Let S_1 denotes the curve fulfilled $y_2 = \psi_1(y_1)$ $(0 \leq y_1 \leq y_1^*)$. Because, for $0 \leq y_1 \leq y_1^*$, $\dfrac{\partial h(y_1,y_2)}{\partial y_2} < 0$; for $y_1^* < y_1 \leq n$, $\dfrac{h(y_1,y_2)}{\partial y_2} \leq 0$, then $D_1 = \{y : 0 \leq y_1 \in < y_1^*,\ \psi_1(y_1) < y_2 \leq n - y_1\}$, $D_2 = \{y : 0 \leq y_2 \leq \psi_1(y_1)\ (0 \leq y_1 < y_1^*),\ 0 \leq y_2 \leq n - y_1\ (y_1^* \leq y_1 \leq n)\}$. By the above demonstration, we obtain that

Lemma 2 *If $\lambda < 1, \lambda\gamma n < S < S^*$, then*

$$r(y_1,y_2) = \begin{cases} y_1 + \lambda\varphi(y_1)[\gamma y_2 + \chi(n - y_1 - y_2)], & \text{for } \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in D_1, \\ S & \text{for } \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in D_2. \end{cases}$$

Lemma 3 *The curve $S_1 : y_2 = \psi_1(y_1)$ $(0 \leq y_1 \leq y_1^*)$ is above the line $S_2 : y_2 = S/\gamma$ $(0 \leq y_1 \leq n - \dfrac{S}{\gamma})$. That is to say, $\psi_1(y_1) > S/\gamma$ $(0 \leq y_1 \leq y_1^*)$.*

Proof First, $\psi_1(0) = \dfrac{\lambda\chi n - S}{\lambda(\chi-\gamma)} \triangleq y_2^*$. Obviously, $y_2^* > \dfrac{\lambda\chi n - S^*}{\lambda(\chi-\gamma)} = \dfrac{\lambda\chi n}{\lambda\chi + (1-\lambda)\gamma} = \dfrac{S^*}{\gamma}$ $> \dfrac{S}{\gamma}$. So, if we demonstrate $\dfrac{d\psi_1(y_1)}{dy_1} \geq 0$ $(0 \leq y_1 \leq y_1^*)$ the Lemma is proved. In fact, if $0 \leq y_1 \leq y_1^*$, then

$$\begin{aligned} \frac{d\psi_1(y_1)}{dy_1} &= \frac{1}{\lambda(\chi-\gamma)\varphi^2(y_1)} \{[1 + \lambda\chi\varphi'(y_1)(n-y_1) - \lambda\chi\varphi(y_1)]\varphi(y_1) \\ &\quad - [y_1 + \lambda\chi\varphi(y_1)(n-y_1) - S]\varphi'(y_1)\} \\ &= \frac{1}{\lambda(\chi-\gamma)\varphi^2(y_1)} [(1 - \lambda\chi\varphi(y_1))\varphi(y_1) + (S - y_1)\varphi'(y_1)]. \end{aligned}$$

From assumption (IV), we obtain

$$\varphi(y_1) = \int_0^{y_1} \varphi'(\tau)d\tau + 1 \geq 1 - \frac{1-\lambda\chi}{\lambda\chi n} y_1.$$

So, if $0 \le y_1 \le y_1^*$, then

$$\begin{aligned}\frac{d\psi_1(y_1)}{dy_1} &\ge \frac{1}{\lambda(\chi-\gamma)\varphi^2(y_1)}[(1-\lambda\chi)(1-\frac{1-\lambda\chi}{\lambda\chi n}y_1)-(S-y_1)\frac{1-\lambda\chi}{\lambda\chi n}]\\ &= \frac{1}{\lambda(\chi-\gamma)\varphi^2(y_1)}[(1-\lambda\chi)(1-\frac{S}{\lambda\chi n})+\frac{1-\lambda\chi}{n}y_1]>0.\end{aligned}$$

In words, $\frac{d\psi_1(y_1)}{dy_1} > 0 \ (0 \le y_1 \le y_1^*)$.

Proposition 1 If $S < \frac{\lambda\chi\gamma n}{\lambda\chi+(1-\lambda)\gamma} = S^*$, then system (2) has exactly one stationary point $\bar{y} \in \operatorname{Int}\bar{Y}$ on $\bar{Y}$.

Proof For $\lambda = 1$ or $\lambda < 1$ and $S \le \lambda\gamma n$, (2) becomes

$$\begin{cases}\frac{dy_1}{dt} = \lambda\varphi(y_1)[\gamma y_2 + \chi(n-y_1-y_2)] - S,\\ \frac{dy_2}{dt} = S - \lambda y_2.\end{cases}$$

In order to find out the stationary point of (2), we may solute equations

$$\begin{cases}\lambda\varphi(y_1)[\gamma y_2 + \chi(n-y_1-y_2)] - S = 0\\ y_2 = S/\gamma.\end{cases}$$

We find easily, $\frac{S}{\gamma} < \frac{S^*}{\gamma} = \frac{\lambda\chi n}{\lambda\chi+(1-\lambda)\gamma} \le n$, So, in order to prove the proposition, we only need to demonstrate that the equation

$$h(y_1) \triangleq \lambda\varphi(y_1)[S + \chi(n - S/\gamma - y_1)] - S = 0$$

has exactly one solution $\bar{y} \in (0, n-S/\gamma)$ on $[0, n-S/\gamma]$. If $\lambda < 1$ and $\lambda\gamma n < S < S^*$. In $\bar{D}_1$, equation (2) is

$$\begin{cases}\frac{dy_1}{dt} = -y_1,\\ \frac{dy_2}{dt} = y_1 + \lambda\varphi(y_1)[\gamma y_2 + \chi(n-y_1-y_2)] - \gamma y_2.\end{cases}$$

The equation has only one stationary point $(0, \bar{y}_2)$ in R^2, where $\bar{y}_2 = \frac{\lambda\chi n}{\lambda\chi+(1-\lambda)\gamma}$. From $\psi_1(0) = y_2^* = \frac{\lambda\chi n - S}{\lambda(\chi-\gamma)} > \frac{\lambda\chi n - S^*}{\lambda(\chi-\gamma)} = \frac{\lambda\chi n}{\lambda\chi+(1-\lambda)\gamma} = \bar{y}_2$, we obtain that, $(0,\bar{y}) \notin \bar{D}_1$, $(0,\bar{y}_2) \in D_2$. That is to say, equation (2) has not any stationary point on $\bar{D}_1$. In D_2, system (2) is

$$\begin{cases}\frac{dy_1}{dt} = \lambda\varphi(y_1)[\gamma y_2 + \chi(n-y_1-y_2)] - S,\\ \frac{dy_2}{dt} = S - \gamma y_2.\end{cases}$$

Searching for the stationary point is equal to solving the following system of equations

$$\begin{cases} \lambda\varphi(y_1)[\gamma y_2 + \chi(n - y_1 - y_2)] - S = 0, \\ y_2 = S/\gamma. \end{cases}$$

From Lemma 3, we know that $S_2 : y_2 = S/\gamma$ $(0 \le y_1 \le n - S/\gamma)$ wholly lies inside D_2. So we only need to demonstrate that the equation $h(y_1) = \lambda\varphi(y_1)[S+\chi(n-S/\gamma-y_1)]-S = 0$ has only one solution $\bar{y}_1 \in (0, n - S/\gamma)$ on $[0, n - S/\gamma]$. In words, for $0 < S < S^*$, proving proposition 1 is equal to proving that the equation $h(y_1) = 0$ has exactly one solution $\bar{y}_1$ on $[0, n - S/\gamma]$ and $\bar{y}_1 \in (0, n - S/\gamma)$.

From $h(0) = \lambda\chi n - \dfrac{\lambda\chi + (1-\lambda)\gamma}{\gamma} S = \dfrac{\lambda\chi + (1-\lambda)\gamma}{\gamma}(S^* - S) > 0$ and $h(n - \dfrac{S}{\gamma}) = -[1 - \lambda\varphi(n - \dfrac{S}{\gamma})]$. $S < 0$, we know, $h(y_1) = 0$ has solutions on $[0, n - S/\gamma]$, and these solutions are inside $(0, n - S/\gamma)$. We assume $\bar{y}_1 \in (0, n - S/\gamma)$ and $h(\bar{y}_1) = 0$. So, from $h(\bar{y}_1) = 0$, we can obtain that $\lambda\varphi(\bar{y}_1)[S + \chi(n - S/\gamma - \bar{y}_1)] = S > 0$. It is thus clear that $\varphi(\bar{y}_1) > 0$. By inspecting, $\dfrac{d}{dy_1}h(y_1) = \lambda\varphi'(y_1)[S + \chi(n - \dfrac{S}{\gamma} - y_1)] - \lambda\chi\varphi(y_1)$. We specially have that, $\dfrac{dh(y_1)}{dy_1}|_{y_1=\bar{y}_1} \le -\lambda\chi\varphi(\bar{y}_1) < 0$. This shows, the solution of the equation $h(y_1) = 0$ on $(0, n - S/\gamma)$ is unique.

Proposition 2 If equation (2) has exactly one stationary point $\bar{y} \in \text{Int}\,\bar{Y}$ on $\bar{Y}$, then $S < S^*$.

Proof If not, then $S \ge S^*$. For $\lambda = 1$ and $S = S^* = \lambda n$, $(0, n)$ is the stationary point of (2) and $(0, n) \in \partial\bar{Y}$, contradiction. For $\lambda = 1, S > S^*$ or $\lambda < 1, S \ge S^*$, in $\bar{D}_1$, equation (2) is

$$\begin{cases} \dfrac{dy_1}{dt} = -y_1, \\ \dfrac{dy_2}{dt} = y_1 + \lambda\varphi(y_1)[\gamma y_2 + \chi(n - y_1 - y_2)] - \gamma y_2. \end{cases}$$

The equation has a stationary point $(0, \bar{y}_2)$ in $\bar{Y}$, where $\bar{y}_2 = \dfrac{\lambda\chi n}{\lambda\chi + (1-\lambda)\gamma}$. Noting that, $\psi_1(0) = y_2^* = \dfrac{\lambda\chi n - S}{\lambda(\chi - \gamma)} \le \dfrac{\lambda\chi n - S^*}{\lambda(\chi - \gamma)} = \dfrac{\lambda\chi n}{\lambda\chi + (1-\lambda)\gamma} = \bar{y}_2$. We know, the stationary point $(0, \bar{y}_2) \in \bar{D}_1$, from $\bar{y}_2 > 0$, we obtain that equation (2) has a stationary point $(0, \bar{y}_2) \in \partial\bar{Y}$ in $\bar{Y}$, contradiction.

Propsition 3 If $S < S^*$, the unique stationary point $\bar{y}$ of equations (2) is asymptctically stable.

Proof From the proof of propostion 1, we know that, if $S < S^*$, $\bar{y}$ is a stationary point of the following equation

$$\begin{cases} \dfrac{dy_1}{dt} = \lambda\varphi(y_1)[\gamma y_2 + \chi(n - y_1 - y_2)] - S \triangleq g_1(y_1, y_2), \\ \dfrac{dy_2}{dt} = S - \gamma y_2 \triangleq g_2(y_1, y_2) \end{cases}$$

and $\bar{y}$ is an interior point of $\bar{Y}$ or D_2. After inspecting that

$$A \triangleq \left.\frac{\partial(g_1,g_2)}{\partial(y_1,y_2)}\right|_{(\bar{y}_1,\bar{y}_2)} = \begin{bmatrix} \frac{\partial}{\partial y_1}g_1(\bar{y}_1,\bar{y}_2) & \frac{\partial}{\partial y_2}g_1(\bar{y}_1,\bar{y}_2) \\ 0 & -\gamma \end{bmatrix},$$

we make out that A has two real eigenvalues,

$$\begin{aligned}\lambda_1 &= -\gamma \\ \lambda_2 &= \frac{\partial}{\partial y_1}g_1(\bar{y}_1,\bar{y}_2) = \lambda\varphi'(\bar{y}_1)[S+\chi(n-\frac{S}{\gamma}-\bar{y}_1)] - \lambda\varphi(\bar{y}_1)\chi.\end{aligned}$$

Clearly, $\lambda_1 = -\gamma < 0$. From assumption (I), we know $\varphi'(\bar{y}_1) \le 0$. From the proof of proposition 1, we know, $\varphi(\bar{y}) > 0$. So, $\lambda_2 \le -\lambda\varphi'(\bar{y}_1)x < 0$. Therefore the stationary point $\bar{y}$ is an asymptotically stable node.

Proposition 4 If $S < S^*$, for every $y_0 \in \bar{Y}$ there exists unique solution $y(t;y_0)$ of system (2) such that $y|_{t=0} = y_0$ and $y(t)$ is defined for all $t \ge 0$ and $y(t;y_0) \in \bar{Y}$ for all $t \ge 0$.

Proof According to assumption (IV), we extend the range of definition of $\varphi(y_1)$ to $(-\delta,+\infty)$ $(\delta > 0$, enough small) such that $\varphi(y_1)$ is monotone non-increasing and continuously differentiable on $(-\delta,+\infty)$. Consequently, the right of system (2) is continuous and satisfy Lipschitz condition on the bounded domain $\Omega = \{y : y \in R^2, -\delta < y_1, y_2 < n+\sigma, y_1+y_2 < n+\delta\}$. So, for every $y_0 \in \Omega$, there exists exactly one solution $y(t;y_0)$ of the system on $[0,\beta(y_0))$, where $[0,\beta(y_0))$ is a maximal interval to which the solution extends right in the domain Ω. Up to now, we only need to prove that for every $y_0 \in \bar{Y}$, $\beta(y_0) = +\infty$ and $y(t;y_0) \in \bar{Y}$ $(t \ge 0)$. Noting that $\bar{Y}$ is a bounded and closed set and $\bar{Y} \subset \Omega$, on the basis of chapter I of Hale (1969) (theorem 2.1), we only need proving that for every $y_0 \in \bar{Y}$, the solution $y(t;y_0) \in \bar{Y}$ $(0 \le t < \beta(y_0))$. Now, we carry out the demonstration by two separate cases.

Case 1 $\lambda = 1$ or $\lambda < 1$ and $S \le \lambda\gamma n$. In $\bar{Y}$, equation (2) becomes

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} f_1(y_1,y_2) \\ f_2(y_1,y_2) \end{pmatrix} = \begin{pmatrix} \lambda\varphi(y_1)[\gamma y_2 + \chi(n-y_1-y_2)] - S \\ S - \gamma y_2 \end{pmatrix}.$$

We might as well regard the equation as one defined in Ω. On L_1, we have $\frac{dy_1}{dt} = \lambda[\gamma y_2 +\chi(n-y_2)] - S > \lambda\gamma n - S \ge 0$ $(0 \le y_2 < n)$, and $\frac{dy_1}{dt} \ge 0$, $\frac{dy_2}{dt} < 0$ at the point $(0,n)$ on L_1. On L_2, $\frac{dy_2}{dt} = S > 0$ $(0 \le y_1 \le n)$. On L_3 the outer normal vector is (1.1), then

$$(1.1)f(y)|_{y_2=n-y_1} = \lambda\varphi(y_1)\gamma(n-y_1) - \gamma(n-y_1) = -(1-\lambda\varphi(y_1))\gamma(n-y_1) < 0,\ 0 < y_1 < n$$

and $(1.1)f(y)|_{(n,0)} = 0$, $(1.1)f(y)|_{(0,n)} \le 0$. In words, on $\partial\bar{Y}$, except point $(0,n)$ and $(n,0)$, for every other point $y_0 \in \partial\bar{Y}$, we have $\delta(y_0) > 0$ such that the solution $y(t;y_0) \in \text{Int}\,\bar{Y}$ $(0 < t \le \delta(y_0))$, $y(t;y_0) \in \Omega/\bar{Y}$ $(-\delta(y_0) \le t < 0)$. Using the continuous dependence of the solution on the initial value, we can prove that for points $(0,n)$ and $(n,0)$, there exists $\delta > 0$ as other points on $\partial\bar{Y}$. So, from the uniqueness of the orbit of system (2), we obtain that, for every $y_0 \in \bar{Y}$, $y(t;y_0) \in \bar{Y}$ $(0 \le t < \beta(y_0))$.

Case 2 $\lambda < 1$, $\lambda\gamma n < S < S^*$. If letting $\delta < \min\{\frac{1}{2}\lambda\chi n\frac{\lambda(\chi-\gamma)}{\lambda\chi+(1-\lambda)\gamma}, \frac{1}{2}\frac{\gamma n}{\chi-\gamma}\}$, we can extend the curve S_1 to the S_1^*:

$$y_2 = \psi_1(y_1) \quad (-\delta < y_1 \le y_1^{**}), \quad \text{where } y_1^* < y_1^{**} < \hat{y}_1,$$

and

$$\psi_1(y_1) = \frac{y_1 + \lambda\varphi(y_1)\chi(n-y_1) - S}{\lambda(\chi-\gamma)\varphi(y_1)},$$

putting $\Omega_1 = \{(y_1,y_2) : \psi_1(y_1) < y_2 < n+\delta-y_1 \ \ (-\delta < y_1 < y_1^{**})\}$, $\Omega_2 = \{(y_1,y_2) : -\delta < y_2 \le \psi_1(y_1) \ \ (-\delta < y_1 < y_1^{**})\}$, $-\delta < y_2 < n+\delta-y_1 \ \ (y_1^{**} \le y_1 < n+\delta)\}$, we have that $\Omega = \Omega_1 \cup \Omega_2$, $D_1 = \Omega_1 \cap \bar{Y}$, $D_2 = \Omega_2 \cap \bar{Y}$. In Ω_1 system (2) becomes that

$$\begin{cases} \dfrac{dy_1}{dt} = -y_1 \\ \dfrac{dy_2}{dt} = y_1 + \lambda\varphi(y_1)[\gamma y_2 + \chi(n-y_1-y_2)] - \gamma y_2; \end{cases}$$

In Ω_2 system (2) becomes that

$$\begin{cases} \dfrac{dy_1}{dt} = \lambda\varphi(y_1)[\gamma y_2 + \chi(n-y_1-y_2)] - S, \\ \dfrac{dy_2}{dt} = S - \gamma y_2 \end{cases}$$

On L_2, we have $\dfrac{dy_2}{dt} = S > 0$ $(0 \le y_1 \le n)$; On L_3, (1.1)$f(y) = -(1-\lambda\varphi(y_1))\gamma(n-y_1) <$ 0, $0 \le y_1 < n$. On L_1, if $0 \le y_2 < y_2^* = \dfrac{\lambda\chi n - S}{\lambda(\chi-\gamma)}$, we have

$$\frac{dy_1}{dt} = \lambda\chi n - S - \lambda(\chi-\gamma)y_2 > 0.$$

Up to now, except for one segment $y_1 = 0$ $(y_2^* \le y_2 \le n)$ and the point $(n,0)$, for every other $y_0 \in \partial\bar{Y}$, there always exists $\delta(y_0) > 0$ such that $y(t;y_0) \in \text{Int}\,\bar{Y}$ $(0 < t \le \delta(y_0))$, $y(t;y_0) \in \Omega/\bar{Y}$ $(-\delta(y_0) \le t < 0)$. For the point $(n,0)$, by using the continuous dependence of the solution on the initial value we can demonstrate that the above-mentioned $\delta > 0$ exists exactly. We point out that $y_1 = 0$ $(y_2^* \le y_2 < n+\delta)$ is the orbit of system (2), and its direction is from $(0, n+\delta)$ to $(0, y_2^*)$. For every $y_2^* < y_2^0 < n+\delta$, the solution of system (2) fulfilling that $y|_{t=0} = \begin{pmatrix} 0 \\ y_2^0 \end{pmatrix}$ is $y_1(t) = 0$, $y_2(t) = (y_2^0 - \bar{y}_2)e^{-[\lambda\chi+(1-\lambda)\gamma]t} + \bar{y}_2$ for $0 \le t \le T(y_2^0) = \dfrac{1}{\lambda\chi+(1-\lambda)\gamma}\ln\dfrac{y_2^0-\bar{y}_2}{y_2^*-\bar{y}_2}$, where $\bar{y}_2 = \dfrac{\lambda\chi n}{\lambda\chi+(1-\lambda)\gamma}$. Up to now orbits starting from every point $(0,y_2)$ on $y_1 = 0$ $(y_2^* \le y_2 < n+\delta)$ all go to the point $(0,y_2^*)$.

Next, we answer how orbits do after passing the point $(0, y_2^*)$. Let us inspect the curve $S_3 : \lambda\varphi(y_1)[\gamma y_2 + \chi(n - y_1 - y_2)] - S = 0$, $0 \le y_1 \le \bar{y}_1$, where $\bar{y} = (\bar{y}_1, \bar{y}_2)$ is a stationary solution of (2) in $\bar{Y}$. Clearly $\varphi(\bar{y}_1) > 0$. So, $\varphi(y_1) > 0 \quad (0 \le y_1 \le \bar{y}_1)$. Therefore, $S_3 : y_2 = \dfrac{\lambda\varphi(y_1)\chi(n - y_1) - S}{\lambda\varphi(y_1)(\chi - \gamma)} \triangleq \psi_2(y_1)$, $0 \le y_1 \le \bar{y}_1$. We find easily, $\psi_2(0) = \psi_1(0) = y_2^*$, $\psi_2(y_1) < \psi_1(y_1) \quad (0 < y_1 \le \bar{y}_1)$. So, $S_3 \subset D_2$, on $S_3 : y_2 = \psi_2(y_1)$, $\quad 0 \le y_1 < \bar{y}_1$, we have $\dfrac{dy_1}{dt} = 0$, $\dfrac{dy_2}{dt} = S - \gamma y_2 < 0$ (This inequality can be obtained from the proof of proposition (1)). By using the continuity of the solution of system (2) to initial value, we can prove that $y(t; \begin{pmatrix} 0 \\ y_2^* \end{pmatrix}) \in D^* \quad (0 < t < \beta(\begin{pmatrix} 0 \\ y_2^* \end{pmatrix}))$, where $D^* = \{(y_1, y_2) : S/\gamma < y_2 < \psi_2(y_1) \quad (0 < y_1 < \bar{y}_1)$ (noting that $y_2 = S/\gamma$, $0 \le y_1 < \bar{y}_1$ is an orbit). In words, from the uniqueness of the orbit we obtain that for every $y_0 \in \bar{Y}$, $y(t : y_0) \in \bar{Y} \quad (0 \le t < \beta(y_0))$.

Proposition 5 If $S < S^*$, for every $y_0 \in \bar{Y}$, $\lim\limits_{t \to \infty} y(t; y_0) = \bar{y}$.

Proof First, we prove system (2) has not any closed orbit in $\bar{Y}$. Otherwise, (2) has a closed orbit $\Gamma \subset \bar{Y}$. From Chapter IV of Sansone and Conti (1964) (Theorem 12), we know that, there must is stationary point in the interior of Γ. From propsition 1 we have that there is only one stationary point $\bar{y}$ in $\bar{Y}$. It is clear that $\bar{y}$ lies in interior of γ. Clearly, $y_2 = \bar{y}_2 = S/\gamma \quad (0 \le y_1 < \bar{y}_1)$ is an orbit of (2). Then the orbit $y_2 = \bar{y}_2 \quad (0 \le y_1 < \bar{y}_1)$ intersects the closed orbit of (2). It contradicts the uniqueness of the orbit. For every $y_0 \in \bar{Y}$, we know from roposition 4 that there exists exactly one solution $y(t; y_0)$ of system (2) and $y(t; y_0) \in \bar{Y}$ for all $t \ge 0$. Since $\bar{Y}$ is a bounded and closed set, we have that $\omega(y_0) \subset \bar{Y}$, where $\omega(y_0)$ is the ω limit set of the positive semiorbit $y(t; y_0) \quad (t \ge 0)$. Noting that there isn't any closed orbit in $\bar{Y}$, from chapter II of Hale (1969) (§1, theorem 1.3), we obtain that $\omega(y_0)$ must include the stationary point. By proposition 1, there exists exactly one stationary point $\bar{y}$ in $\bar{Y}$. So, $\bar{y} \in \omega(y_0)$. If $\omega(y_0)$ includes a ordinary point y^+ except the stationary point, then by the above—mentioned theorem of Hale (1969), we know that there exist $\lim\limits_{t \to +\infty} y(t; y^+)$ and $\lim\limits_{t \to -\infty} y(t; y^+)$ and both of them are stationary points in $\omega(y_0)$. So, $\lim\limits_{t \to +\infty} y(t; y^+) = \lim\limits_{t \to -\infty} y(t; y^+) = \bar{y}$. From proposition 2 we know that $\bar{y}$ is a stable node. It contradicts $\lim\limits_{t \to -\infty} y(t; y^+) = \bar{y}$. Clearly, $\omega(y_0)$ only consists of the stationary point. That is to say $\omega(y_0) = \{\bar{y}\}$. Now we prove that $\lim\limits_{t \to \infty} y(t; y_0) = \bar{y}$. If not, there exists $\varepsilon_0 > 0, t_n \uparrow \infty \quad (n \to \infty)$ such that $|y(t_n; y_0) - \bar{y}| \ge \varepsilon_0$. Because $\bar{y}$ is a ω limit point of postive semiorbit $y(t; y_0) \quad (t \ge 0)$, we have that $t'_n \to \infty \quad (n \to \infty)$ such that $|y(t_n^*; y_0) - y| < \varepsilon_0$, and $t_n^* > t_n$. From the continuity of $|y(t; y_0) - \bar{y}|$ we know that there is $t_n < t_n^* < t'_n$ such that $|y(t_n^*; y_0) - \bar{y}| = \varepsilon_0$. Then the sequsnce $\{y(t_n^*; y_0)\}$ has convergent subsequences. We might as well assume the sequence itself converges to $\bar{y}'$. So we have $t^* \to \infty \quad (n \to \infty)$ such that $\lim y(t_n^*; y_0) = \bar{y}'$. Clearly, $\bar{y}' \in \omega(y_0)$. We note $|\bar{y}' - \bar{y}| = \lim\limits_{n \to \infty} |y(t_n^*; y_0) - \bar{y}| = \varepsilon_0$. We have $\bar{y}' \ne \bar{y}$. It contradicts $\omega(y_0) = \{\bar{y}\}$.

The theorem of Chapter I follows from proposition 1—5.

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Kornai-Weibull 排队模型的纰漏，修正与分析

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摘　要

本文首先指出Kornai-Weibull 排队模型(A) 中之接待率$r(y_1)$ 的定义是不正确的，并给出接待率$r(y_1, y_2)$ 的正确定义. 其次，证明了修正模型(1) 有“正常状态” 的充要条件是接待能力

$$S < S^* \triangleq \frac{\lambda\gamma\chi n}{\lambda\chi + (1-\lambda)\gamma}.$$

于是，本文发现接待能力S 的分歧值是S^*; 当$S < S^*$ 时，市场是短缺的; 当$S \geq S^*$ 时，市场是不短缺的.