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带有动力学条件的高维一相Stefan问题

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摘　要

本文我们证明了带有动力学条件的高维一相Stefan问题局部古典解的存在唯一性。

Journal of Mathematical Research & Exposition
Vol.14, No.4, Nov. 1994

The Multi-Dimension One Phase Stefan Problem with Kinetic Condition *

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Abstract In this paper we prove that there exists a unique local classical solution for the multidimension one phase Stefan problem with Kinetic condition.

Keywords free boundary problems, stefan problem, kinetic condition.

1. Introduc

We consider a material which may be in either of two phases, e.g., solid and liquid, occupying a region $\Omega \in \mathcal{R}^N$. Let u denote the temperature and for each time $t > 0$, denote by Γ_t the hypersurface which separate the solid region Ω_t^1 and the liquid region Ω_t^2. The classical two phase Stefan problem is to find a pair (u, Γ) satisfying

$$u_t^i - k^i \triangle u^i = 0 \qquad \text{in } Q^i \equiv \cup_t \Omega_t^i \quad (i = 1, 2), \tag{1.1}$$

$$k^1 \frac{\partial u^1}{\partial n} - k^2 \frac{\partial u^2}{\partial n} = LV_n \qquad \text{on } \Gamma \equiv \cup_t \Gamma_t, \tag{1.2}$$

$$u^1 = u^2 = 0 \qquad \text{on } \Gamma \tag{1.3}$$

with suitable initial and boundary conditions, where $u = u^i$ in Q^i, k^i are the thermal diffusivity coefficients, L is the latent heat, n the normal to Γ_t, and V_n is the normal velocity of the hypersurface Γ_t.

This problem has been extensively studied as a model for phase transitions, see, for instance, [1]. However, the physical situation is generally more complex than that described by the classical Stefan formulation. A prevailing alternative formulation (see [2] and its references) which attempts to recover these effects is obtained by replacing the condition of u being equal to the equilibrium melting temperature on the interface, by the "modified Gibbs- Thomsono relation"

$$u = -\sigma k - \beta V_n \quad \text{on} \quad \Gamma. \tag{1.4}$$

Here k denotes the sum of the principal curvatures at a point on the interface, α and β are non-negative constants.

*Received Sep. 15, 1992. Supported by the National Science Foundation of China.

When $\sigma > 0$ and $\beta > 0$, problem (1.1),(1.2), (1.4) have an unique local classical solution ([3]). When $\beta = 0$, problem (1.1), (1.2), (1.4) have global weak solution ([4]), and local classical solution ([5]). For these two cases the condition (1.4) is an equation of second order. But, for $\sigma = 0$, no result about the existence of the classical solution to problem (1.1), (1.2), (1.4) is known for the multi-dimension case, since the condition (1.4) at this time becomes an equation of first order on a manifold, which causes new difficulty for the regularity to the space variable of the free boundary. This paper is devoted to solve this problem for one phase. We establish that there exists an unique local classical solution, which provide a parabolic analog of the result in [6]. Precisely we consider following free boundary problem:

Find a function $u = u(x, y, t)$ and a surface $\Gamma : y = g(x, t) \quad (x, t) \in \mathcal{R}^1 \times [0, T]$ satisfy

$$u_t - \triangle u = 0 \qquad \text{in } D_T \equiv \cup_{t\in[0,T]}\Omega_t, \tag{1.5}$$

$$u(x, 0, t) = b(x, t) \qquad \text{on } \tau \equiv \mathcal{R}^1 \times [0, T], \tag{1.6}$$

$$u(x, y, 0) = u_0(x, y) \qquad \text{on } \Gamma_0, \tag{1.7}$$

$$\frac{\partial u}{\partial n} + u = 0 \qquad \text{on } \Gamma \equiv \cup_t\Gamma_t, \tag{1.8}$$

$$u = V_n \qquad \text{on } \Gamma, \tag{1.9}$$

and g satisfies the initial condition

$$g(x, 0) = g_0(x) > 0, \quad \forall x \in \mathcal{R}^1. \tag{1.10}$$

Here $\Omega_t \equiv \{(x, y)|x \in \mathcal{R}^1, \quad 0 < y < g(x, t)\}$, $\Gamma_t \equiv \{(x, g(x, t))|x \in \mathcal{R}^1\}$, n is the outward normal to Γ_t for each $t \in [0, T]$ fixed, and V_n is the velocity of the free boundary $V_n = \dfrac{g_t}{\sqrt{1+g_x^2}}$.

We shall assume:

$$u_0(x, 0) = b(x, 0), \qquad \frac{\partial b}{\partial t}(x, 0) - \triangle u_0(x, 0) = 0 \quad \forall x \in \mathcal{R}, \tag{1.11}$$

$$\frac{\partial u_0}{\partial n} + u_0 = 0 \qquad \text{on } \Gamma_0, \tag{1.12}$$

where n is the outward normal to Γ_0, (1.11) and (1.12) are the compatibility condition.

$$u_0(x, y) \in C^{2+\alpha}(\bar{\Omega}_0), \quad \|u_0\|_{C^{2+\alpha}(\bar{\Omega}_0)} \le c_1, \tag{1.13}$$

$$g_0(x) \ge c_0 > 0, \quad \|g_0\|_{C^{2+\alpha}(\mathcal{R}^1)} \le K, \tag{1.14}$$

$$b(x, t) \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\tau), \quad \|b\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\tau)} \le c_1, \tag{1.15}$$

where c_0 and K are given constants, c_1 is a given constant independent of T. Our main result is the following:

Theorem *Under the assumption (1.11)—(1.15), problem (1.5)— (1.10) has one and only one local classical solution.*

2. Existence

Define the class of functions

$$B_{M,M_1}=\{g(x,t),\ \ x\in\mathcal{R}^1,\ \ 0\le t\le T;\ \ g(x,t)\ge\frac{c_0}{2},\ \ \|g\|_{C^{2+\alpha,0}_{x,t}(\tau)}\le 8K,$$
$$\|g_t\|_{L^\infty(\tau)}\le M,\ \ [g_t]_{C^{\alpha,\frac{\alpha}{2}}_{x,t}(\tau)}\le M_1,\ \ g(x,0)=g_0(x)\},$$

where M and M_1 are positive constants to be determined.

For any $g\in B_{M,M_1}$ define $\Gamma: y=g(x,t))$ $((x,t)\in\tau)$ and let u be the bounded solution of (1.5)—(1.8). By the maximum principle

$$\|u\|_{L^\infty(D_T)}\le c_1. \tag{2.1}$$

Choose $M=\sqrt{1+64K^2}c_1+K$. By the Krylov estimates

$$\|u\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{D}_T)}\le c_2. \tag{2.2}$$

Here c_2 depends on K,c_1,M,T, but not on the lower bound of T. So $c_2=c_2(K,c_1)$. Now choose $M_1=2^8Kc_2$. By Schauder estimates (see [8], Th. 11.6; or [9], Th. IV. 5.3)

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}_T)}\le c_3=c_3(K,c_1,M,M_1,T). \tag{2.3}$$

It is easy to see that c_3 is independent of the lower bound of T. So c_3 depends only on K,c_1.

Let φ_δ be mollifiers in x and set $V_\delta(x,t)=(\varphi_\delta*v(\cdot,t))(x)$ where $v(x,t)=u(x,g(x,t),t)$. Then

$$\|V_\delta(\cdot,t)\|_{C^{2+\alpha}(\mathcal{R}^1)}\le c_3,\ \ \|V_\delta\|_{L^\infty(\tau)}\le c_1,\ \ \|V_\delta\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{D}_T)}\le c_2. \tag{2.4}$$

Introduce $g_\delta(x)=\varphi_\delta*g_0(x)$, we have

$$\|g_\delta\|_{C^{2+\alpha}(\mathcal{R}^1)}\le K. \tag{2.5}$$

For any small $\varepsilon>0$, let $\tilde{g}(x,t)$ be the bounded solution of

$$\tilde{g}_t=\sqrt{1+\tilde{g}_x^2}V_\delta(x,t)+\varepsilon\tilde{g}_{xx}, \tag{2.6}$$
$$\tilde{g}(x,0)=g_\delta(x). \tag{2.7}$$

By comparison [10, pp.52]

$$\tilde{g}(x,t)\le c_1t+\max g_\delta(x),\ \ \tilde{g}(x,t)\ge -c_1t+\inf g_\delta(x), \tag{2.8}$$

so that

$$\tilde{g}(x,t)>\frac{c_0}{2},\ \ \|\tilde{g}\|_{L^\infty(\tau)}\le 2K, \tag{2.9}$$

T is small enough (depending only on c_1,K).

The following arguments are the almost same as in [6]. In order to render the paper as self-contained as possible we will reproduce certain calculations already presented in [6].

Differentiate (2.6) in x to obtain

$$L\tilde{g}_x \equiv \frac{\partial}{\partial t}\tilde{g}_x - \frac{V_\delta}{\sqrt{1+\tilde{g}_x^2}}\tilde{g}_x - V_{\delta,x}\sqrt{1+\tilde{g}_x^2} - \varepsilon\frac{\partial^2}{\partial x^2}\tilde{g}_x. \tag{2.10}$$

The function $w = ct + K$ satisfies $Lw = c - V_{\delta,x}\sqrt{1+w^2} > 0$ if $c \geq c_k$ and T small depending only on K, c_1. By comparison with $\tilde{g}_x$ we get

$$\tilde{g}_x \leq w \leq 2K, \tag{2.11}$$

if T is small enough, and similarly

$$\tilde{g}_x \geq -2K. \tag{2.12}$$

Next differentiating (2.6) in x twice and using (2.9), (2.1), (2.13), we obtain by comparison, as before,

$$|\tilde{g}_{xx}| \leq ct + K \leq 2K, \tag{2.13}$$

where $c = c(K, c_1)$ and T is small depending only on K, c_1.

Finally, from (2.6)

$$|\tilde{g}_t|_{L^\infty(\tau)} \leq \tilde{M} \equiv \sqrt{1+4K^2}c_1 + K, \tag{2.14}$$

if $\varepsilon \leq 1/2$ and T is small. Also from (2.2), (2.4).

V_δ is Hölder continuous in (x,t), $\|V_\delta\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{D}_T)} \leq c_2$.

Next we observe that the problem

$$g_t = \sqrt{1+g_x^2}V_\delta(x,t), \tag{2.16}$$

$$g(x,0) = g_\delta(x) \tag{2.17}$$

has at most one solution. In fact, this follows by estimating the difference of two solutions, making use of the Lipschitz continuity of $V_\delta(x,t)$ in x and its continuity in t.

From the above observations and the estimates (2.9), (2.11), (2.12), (2.13), (2.14) it follows that the family $\tilde{g} \equiv \tilde{g}_\varepsilon$ converges to a (unique) solution g^* of (2.16), (2.17) as $\varepsilon \to 0$.

By using (2.4), (2.13), (2.16) we get

$$\|g_t^*\|_{L^\infty(\tau)} \leq M, \quad [g_t^*]_{C^{\alpha,\frac{\alpha}{2}}(\tau)} \leq M_1.$$

Now differentiating (2.16) formally twice in x to obtain

$$\frac{\partial}{\partial t}g_{xx}^* - \frac{V_\delta g_x^*}{\sqrt{1+g_x^{*2}}}g_{xxx}^* = \frac{(g_{xx}^*)^2 V_\delta}{(1+g_x^{*2})^{3/2}} + \frac{2g_x^* g_{xx}^*}{\sqrt{1+g_x^{*2}}}V_{\delta,x} + \sqrt{1+g_x^{*2}}V_{\delta,xx}. \tag{2.18}$$

To justify this differentiation note that by differentiating (2.10) successively in x and comprising with function of the form $ct + \tilde{c}$ we can estimate the derivatives $\tilde{g}_{xxx}, \tilde{g}_{xxxx}$, etc., as we have done in (2.13). The constants depend on δ but not on ε. Hence by differentiating (2.6) twice in x and then letting $\varepsilon \to 0$, equation (2.18) follows.

Next introduce the characteristics

$$\frac{d\xi}{dt} = -(\frac{g_x^*}{\sqrt{1+g_x^{*2}}} V_\delta)(\xi,t), \quad \xi(x,0)=x \tag{2.19}$$

and note that $\frac{1}{2} \le \frac{d\xi}{dx} \le 2$ if T is small. Writing (2.18) in integrated form along characteristics we can conclude that

$$\begin{aligned} |g_{xx}^*(\xi(x_1,t),t) - |g_{xx}^*(\xi(x_x,t),t)| &\le |g_{\delta,xx}(x_1) - g_{\delta,xx}(x_2)| \\ &+ \int_0^t |A_1[g_{xx}^*(\xi(x_1,s),s) - g_{xx}^*(\xi(x_2,s),s)] + A_2|ds, \end{aligned}$$

where $|A_j| \le c_K \ \ (j=1,2)$, $|A_2| \le c_K|\xi(x_1,s)-\xi(x_2,s)|^\alpha$, c_K independent of δ.

It is easily follows from Gronwall inequality that $[g_{xx}^*]_{C_x^\alpha} \le 2K$ if T is small.

Consider the mapping W defined by $g \to Wg = g^*$. We have proved that W maps B_{M,M_1} into itself provided T is sufficiently small (depending only on K, c_1, but not on δ).

If we provide B_{M,M_1} with the uniform topology, then B_{M,M_1} is compact and convex subset. From the uniqueness of solution to (2.16), (2.17) and compactness if follows that W is continuous. By Schauder fixed-point theorem, W has a fixed point g_δ^*. Taking $\delta \to 0$ through an appropriate subsequence we obtain a limiting function g, which together with the corresponding u, provide a local classical solution to (1.5)—(1.10).

3. Uniqueness

In 2. we have proved the existence of solutions (u,g) such that

$$g \in C^{2+\alpha,1+\frac{\alpha}{2}}(\tau). \tag{3.1}$$

Now assume that for some $T>0$, $(u,g),(\tilde{u},\tilde{g})$ are two solutions of (1.5)—(1.10) satisfying (3.1). We shall prove $u=\tilde{u}$ and $g=\tilde{g}$.

By assumption, $||g||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\tau)} ||\tilde{g}||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\tilde{\tau})} \le c$ and therefore by Schauder's estimates

$$||u||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{D}_T)} \le c, \quad ||\tilde{u}||_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\tilde{D}}_T)} \le c. \tag{3.2}$$

Put

$$V(t) = \sup_x |g(x,t) - \tilde{g}(x,t)| \quad (t \text{ small }), \tag{3.3}$$

and denote $G_t = \{(x,y)|0<y<g(x,t)-V(t)\}$ and $S_t = \{y = g(x,t)-V(t)\}$. Then $\partial G_t = \{y=0\} \cup S_t$ and the outward normal along S_t is $n = \frac{(-g_x,1)}{\sqrt{1+g_x^{*2}}}$. Set

$$\begin{aligned} J_1 &= (\frac{\partial u}{\partial n} + u)|_{y=g(x,t)-V(t)} - (\frac{\partial u}{\partial n} + u)|_{y=g(x,t)}, \\ J_2 &= (\frac{\partial \tilde{u}}{\partial n} + \tilde{u})|_{y=g(x,t)-V(t)} - (\frac{\partial \tilde{u}}{\partial n} + \tilde{u})|_{y=\tilde{g}(x,t)}. \end{aligned}$$

Then, by (3.2)

$$\|J_1\|_{C_x^\alpha(\mathcal{R}^1)} \le cV(t), \tag{3.4}$$

$$\|J_2\|_{C_x^\alpha(\mathcal{R}^1)} \le cV(t) + c\|g(\cdot,t) - \tilde{g}(\cdot,t)\|_{C_x^\alpha(\mathcal{R}^1)}. \tag{3.5}$$

Introducing the normal to $\tilde{\Gamma}_t = \{y = \tilde{g}(x,t)\}$, $\tilde{n} = \dfrac{(-\tilde{g}_x, 1)}{\sqrt{1+\tilde{g}_x^{*2}}}$. We also have

$$\|(\frac{\partial \tilde{u}}{\partial n} - \frac{\partial \tilde{u}}{\partial \tilde{n}})|_{y=\tilde{g}(x,t)-V(t)}\|_{C_x^\alpha(\mathcal{R}^1)} \le \|g_x(\cdot,t) - \tilde{g}_x(\cdot,t)\|_{C_x^\alpha(\mathcal{R}^1)}. \tag{3.6}$$

From (1.8) for both u and $\tilde{u}$ and (3.4)—(3.6) it easily follows

$$\|(\frac{\partial(u-\tilde{u})}{\partial n} + (u-\tilde{u}))_{y=\tilde{g}(x,t)-V(t)}\|_{C_x^\alpha(\mathcal{R}^1)} \le c\|g-\tilde{g}\|_{C_x^{1+\alpha}(\mathcal{R}^1)}(t) \tag{3.7}$$

and then by maximum principle

$$\|u-\tilde{u}\|_{L^\infty(\cup_{0\le t\le t_0} G_t)} \le c \max_{0\le t\le t_0} \|g_x - \tilde{g}_x\|_{C_x^{1+\alpha}(\mathcal{R}^1)}(t). \tag{3.8}$$

So by parabolic $C^{1+\alpha,1+\frac{\alpha}{2}}$ estimates [17, Th 6.27],

$$\|u-\tilde{u}\|_{C^{2+\alpha,1+\frac{1+\alpha}{2}}\overline{(\cup_{0\le t\le t_0} G_t)}} \le c \max_{0\le t\le t_0} \|g-\tilde{g}\|_{C_x^{1+\alpha}(\mathcal{R}^1)}(t). \tag{3.9}$$

Here c is independent of the lower bound of t_0. Next, differentiating in x the free boundary condition $g_t = \sqrt{1+g_x^2}u(x,g,t)$ we get

$$g_{xt} = H(x,t)g_{xx} + K(x,t), \tag{3.10}$$

where

$$\begin{aligned} H(x,t) &= \frac{g_x}{\sqrt{1+g_x^2}}(u(x,g(x,t),t), \\ K(x,t) &= \sqrt{1+g_x^2}(u_x(x,g(x,t),t) + u_y(x,g(x,t),t)g_x). \end{aligned}$$

A similar formula holds for $\tilde{g}$.

Using (3.2) and (3.9) we can estimate

$$\|K-\tilde{K}\|_{C_x^\alpha(\mathcal{R}^1)}(t) + \|H-\tilde{H}\tilde{g}_{xx}\|_{C_x^{1+\alpha}(\mathcal{R}^1)}(t) \le c \max_{0\le s\le t} \|g(\cdot,s) - \tilde{g}(\cdot,s)\|_{C_x^{1+\alpha}(\mathcal{R}^1)}.$$

Consequently the function $g^* = g - \tilde{g}$ satisfies

$$\|g_{xt}^* - H(x,t)g_{xx}^*\|_{C_x^\alpha}(t) \le c \max_{0\le s\le t} \|g^*\|_{C_x^{1+\alpha}(\mathcal{R}^1)}(s), \tag{3.11}$$

$$\|H(\cdot,t)\|_{C_x^{1+\alpha}(\mathcal{R}^1)} \le c. \tag{3.12}$$

We introduce the characteristics

$$\frac{d\xi}{dt} = -H(\xi,t), \quad \xi(x,0) = x. \tag{3.13}$$

By (3.12)

$$\frac{1}{2} \leq \frac{d\xi}{dt} \leq 2 \quad \text{for} \quad 0 < t < t_0 \tag{3.14}$$

if t_0 is small enough.

Integrating (3.11) along characteristics we obtain

$$\|g_x^*\|_{L_x^\infty}(t) \leq ct \max_{0\leq s\leq t} \|g^*\|_{C_x^{1+\alpha}}(s). \tag{3.15}$$

Using (3.14) and proceeding as in 2. we can also get the estimate

$$\|g_x^*\|_{C_x^{1+\alpha}}(t) \leq ct \max_{0\leq s\leq t} \|g^*\|_{C_x^{1+\alpha}}(s). \tag{3.16}$$

From (3.15), (3.16) and $g_t = \sqrt{1+g_x^2}u(x,g,t)$ it follows that

$$\|g^*\|_{C_x^{1+\alpha}}(t) \leq ct \max_{0\leq s\leq t} \|g^*\|_{C_x^{1+\alpha}}(s).$$

So $g(x,t) \equiv \tilde{g}(x,t)$ for $0 \leq t \leq t_0$, if t_0 is sufficiently small, and then also $u(x,y,t) \equiv \tilde{u}(x,y,t)$ for $0 \leq t \leq t_0$. Now we can proceed step-by-step to prove that $g = \tilde{g}$, $u = \tilde{u}$ for all $0 \leq t \leq T$.

Remark The method of this paper can be used to prove the same result for the multi-dimension problem case.

Acknowledgement *The authors heartily thanks Prof. Jiang Lishang for his guidance and all the other colleagnes in the P.D.E. seminar of Suzhou University for their help.*

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带有动力学条件的高维一相Stefan问题

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摘　要

本文我们证明了带有动力学条件的高维一相Stefan 问题局部古典解的存在唯一性。