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算子点态遍历定理

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摘　要

本文证明了关于$L^p(1 \leq p \leq \infty)$空间中线性算子与非线性算子的几个点态遍历定理，还推广了Yosida与Kakutani关于拟紧算子的一致遍历定理.

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On Pointwise Ergodicity of Mappings in L^p Space *

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Abstrict In this note we prove several theorems on pointwise ergodicity of mappings defined in $L^p(1 \le p \le \infty)$ and generalize the Yosida-Kakutani theorem on uniform ergodicity of quasi-compact operators.

Keywords linear operator, non-linear operator, pole of an operator, pointwise ergodicity.

1. Introduction

Let E be a non-zero complex Banach space, and F, a closed convex subset of E. Let A denote a mapping of F into itself. Write

$$A_n = \frac{1}{n}\sum_{i=0}^{n-1} A^i, \quad n = 1, 2, \cdots. \tag{1}$$

If for any element $x \in F$, the sequence $\{A_n x\}$ converges weakly or strongly, A is called to be weakly or strongly ergodic respectively; In case $E = L^p(S, \Sigma, \mu)$ and the limit

$$\lim_{n\to\infty} A_n x \quad a.e. \tag{2}$$

exists for each element x in F, A is said to be pointwise ergodic (p.e.) or to have pointwise ergodic property (p.e.p.).We are mainly concerned in this note with conditions under which a linear contraction A, i.e. $\|A\| \le 1$, defined on L^p is p.e.([1], [3], [4]). So far as we know sufficient conditions have been obtained as follows: Suppose A is a linear contraction on L^p. (a) If A is positive, i.e., $Ax \ge 0$ for $x \in L^p$ with $x \ge 0$, and $1 < p < \infty$, then A is p.e.; if $p = 1$ or $p = \infty$, the answer is negative. (b) If A is a linear contraction for every p with $1 \le p \le \infty$, $p \ne 2$, A has p.e.p.. (c) If A is convertibly norm-preserving and $1 < p < \infty$, $p \ne 2$, A is p.e.; if $p = 2$, the answer is negative. Therefore the problem is

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still open for general linear contractions. As for nonlinear contractions, it is more difficult and challenge.

In this note some interesting results are obtained. We give a sufficient condition for a nonlinear mapping defined in L^p to be p.e. and then prove that well known contractions of several type have p.e.p..

For simplicity, $\mathcal{B}(E)$ always denotes the Banach algebra of all the bounded linear operators on Banach space E. For $A \in \mathcal{B}(E)$, $\sigma(A)$ represents the spectrum of A, $\gamma(A)$, the spectral radius of A. A complex λ_0 is called a pole of A if λ_0 is both an isolated point of $\sigma(A)$ and a pole of $(\lambda I - A)^{-1}$. Simple poles refer to ones with order one. $L^p(S, \Sigma, \mu)$ space is often abbreviated as L^p.

2. Theorems on Pointwise ergodicity for Mappings

Lemma 1 *Let F be a closed convex subset of Banach space E and A, a mapping of F into itself. If $x_0 \in F$ satisfies*

$$\sum_{k=1}^{\infty} \frac{1}{k+1} \|A_k x_0 - A^k x_0\| < \infty, \tag{3}$$

then (a) $\{A_n x_0\}$ converges strongly; (b) In the case of $E = L^p (1 \leq p \leq \infty)$, $\{A_n x_0\}$ converges a.e. as well.

Proof (a) Since $A_n x_0 - A_{n+1} x_0 = \frac{1}{n+1}(A_n x_0 - A^n x_0)$, $n = 1, 2, \cdots$, we have

$$x_0 - A_{n+1} x_0 = \sum_{k=1}^{n} (A_k x_0 - A_{k+1} x_0) = \sum_{k=1}^{n} \frac{1}{k+1} (A_k x_0 - A^k x_0). \tag{4}$$

The desired conclusion follows from (3) and (4).

(b) Case of $p = 1$. Now Condition (3) takes the form

$$\sum_{k=1}^{\infty} \frac{1}{k+1} \int_S |A_k x_0(s) - A^k x_0(s)| d\mu(s) < \infty.$$

From the monotone converging theorem, we have

$$\int_S \sum_{k=1}^{\infty} \frac{1}{k+1} |A_k x_0(s) - A^k x_0(s)| d\mu < \infty,$$

and hence $\sum_{k=1}^{\infty} \frac{1}{k+1} (A_k x_0(s) - A^k x_0(s))$ converges a.e. to an element of L^1, say, $y^*(s)$. Then (4) implies $\{A_n x_0(s)\}$ tends a.e. to $x_0(s) - y^*(s)$.

Case of $1 < p < \infty$. Let $\frac{1}{p} + \frac{1}{q} = 1$. If $\mu(S) < \infty$, from the Hölder inequality

$$\int_S |x| d\mu \leq \|x\|_p (\mu(S))^{\frac{1}{q}}, \quad \forall x \in L^p$$

and (3), we infer that

$$\sum_{k=1}^{\infty} \frac{1}{k+1} \int_S |A_k x_0 - A^k x_0| d\mu \le (\mu(S))^{\frac{1}{q}} \sum_{k=1}^{\infty} \frac{1}{k+1} ||A_k x_0 - A^k x_0||_p < \infty.$$

Hence, it can be shown in a similar way that $\{A_n x_0(s)\}$ converges a.e. and so does in case S is of σ-finite measure. When S is of non-σ-finite measure, there exists a measurable set S_0 of σ-finite measure such that every $A_n x_0$ vaniches on the complement of S_0. Inplacing S by S_0 and repeating the argument above, one can see that $\{A_n x_0\}$ converges a.e. on S_0 and hence on S.

Case of $p = +\infty$. Obviuos. □

Theorem 2 *Let F be a closed convex subset of L^p with $1 \le p \le \infty$, and let A be a mapping of F into itself satisfying one of the following conditions: (a) There exists a positive integer m such that A^m is a proper contraction. (b) There exists a constant h with $0 < h < 1$ such that for each pair of x, y in F,*

$$||Ax - Ay|| \le h \max\{||x - Ax||, ||y - Ay||, ||x - y||\}.$$

Then A is pointwise ergodic.

Proof Ommitted.

3. Theorems on Pointwise Ergodicity for Bounded Linear Operators

We consider in this section bounded linear operators on a complex Banach space E. We say that $A \in \mathcal{B}(E)$ is uniformly ergodic, if $\{A_n\}$ converges in the uniform operator topology.

We first note that it is easily seen, by a similar argument used as in the proof of Lemma 1, that if $A \in \mathcal{B}(E)$ satisfies the condition: $\sum_{n=1}^{\infty} \frac{1}{n} ||A_n - A^n|| < \infty$, then A has uniformly ergodic property. This fact will be applied in the proof of Lemma 3 below.

Lemma 3 *If the spectral radius of A in $\mathcal{B}(E)$ is strictly less than 1, A is uniformly ergodic. Moreover, in the case of $E = L^p$ with $1 \le p \le \infty$, A is p.e. as well.*

Proof We obtain from elementary properties of bounded linear operators that both $\sum_{n=1}^{\infty} \frac{1}{n} ||A_n|| < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{n} ||A^n|| < \infty$ hold and so does $\sum_{n=1}^{\infty} \frac{1}{n} ||A_n - A^n|| < \infty$. Thus A is uniformly ergodic by the remark before this lemma. Further, take a positive number δ and a positive integer m so that $\gamma(A) < \delta < 1$ and $||A^m|| < \delta^m$. Therefore A satisfies Condition (a) in Theorem 2, and hence A has p.e.p.. □

Lemma 4 *Suppose A and B in $\mathcal{B}(E)$ are commutative, i.e., $AB = BA$. If both A and B are strongly (or uniformly) ergodic, so is $A + B$. Moreover, in the case of $E = L^p$, if both A and B have p.e.p., so does $A + B$.*

Proof Obvious. □

Theorem 5 *Let $A \in \mathcal{B}(E)$. Suppose the spectrum $\sigma(A)$ of A consists of two disjoint sets σ_1 and σ_2 such that $\sigma_1 \subset \{\lambda : |\lambda| < \delta\}$ with $\delta < 1$ and $\sigma_2 \subset \{\lambda : |\lambda| = 1\}$. Put $P_1 = \frac{1}{2\pi i}\int_{|\lambda|=\delta}(\lambda I - A)^{-1}d\lambda$ and $P_2 = I - P_1$. Then*

(a) A has uniformly or strongly ergodic property iff AP_2 does.

(b) In case $E = L^P$ with $1 \le p \le \infty$, it is also true that A is p.e. iff AP_2 is.

Proof According to the spectral theory of bounded linear operators on a complex Banach space we see that $A = AP_1 + AP_2$ and that the spectrum of A_i, the restriction of A to $E_i = P_iE$ is $\sigma_i(i = 1,2)$. Since $AP_1 \cdot AP_2 = AP_2 \cdot AP_1$, the desired conclusions follow from Lemmas 3 and 4 above. □

Lemma 6 *Let $A \in \mathcal{B}(E)$. Each pole of A on the unit circle is simple if one of th following statements holds:*

(a) A is weakly ergodic.

(b) $\left\{\frac{A^n}{n}\right\}$ converges to zero weakly.

(c) A is p.e. in the case of $E = L^p$ with $1 \le p \le \infty$.

Proof (a) Let λ be a pole of A with $|\lambda| = 1$. Assume that λ is not a simple pole. Then there be two non-zero elements x and y in E such that both $(A-\lambda I)x = y$ and $(A-\lambda I)y = 0$ hold([1], p.709). It follows by induction that $A^n x = \lambda^n x + n\lambda^{n-1}y$, $n = 1, 2, \cdots$, and hence

$$A_n x - \frac{1}{n}x = \frac{1}{n}\sum_{k=1}^{n-1} A^k x = \alpha_n x + \beta_n y, \qquad (\Delta)$$

where $\alpha_n = \frac{1}{n}\sum_{k=1}^{n-1}\lambda^k$, $\beta_n = \frac{1}{n}\sum_{k=1}^{n-1}k\lambda^{k-1}$. Observe that $\{\alpha_n\}$ is convergent, while $\{\beta_n\}$ is not.On the other hand, for $\phi \in E^*$ with $\phi(y) \ne 0$, it follows from (a) that $\{\phi(A_n x)\}$ converges weakly. Then we see from (Δ) that $\{\beta_n\}$ should converge also, a contraduction.

(c) It can be shown in a similar way as above.

(b) This is nothing, but Lemma 1 in ([1], p.709). □

Lemma 7 *Let A be an element in $\mathcal{B}(E)$ such that $\sigma(A) \subset \{\lambda : |\lambda| = 1\}$ and each λ in $\sigma(A)$ is a pole of A. Then the following statements are equivalent:*

(a) A is uniformly ergodic.

(b) A is weakly ergodic.

(c) $\left\{\frac{A^n}{n}\right\}$ converges weakly to zero.

(d) each pole of A is simple.

(e) In the case of $E = L^p$ with $1 \le p \le \infty$, A is p.e.

Proof By Lemma 6 above we see that $(a) \Rightarrow (b) \Rightarrow (d)$, $(c) \Rightarrow (d)$, and $(e) \Rightarrow (d)$ are all valid. We have only to show that $(d) \Rightarrow (a) \Rightarrow (c)$ and $(d) \Rightarrow (e)$ are also true.

$(d) \Rightarrow (a)$. Since each pole of A is an isolated point of $\sigma(A)$, $\sigma(A)$ consists of finite point, say, $\sigma(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_m\}$, and each λ_i is a simple pole. Therefore there exist m bounded projections $P_1, P_2, \cdots, P_m$ such that $\sum_{i=1}^{m} P_i = I, \quad P_i \cdot P_j = 0, \quad i \neq j$ and each $E_i = P_i E$ is the eigenspace of A corresponding to λ_i(see [1],V I I.3). Then for any x in E, we have

$$A_n x = \sum_{j=1}^{m} \frac{1}{n} \sum_{i=0}^{n-1} {\lambda_j}^i P_j x. \tag{5}$$

If $1 \in \sigma(A)$ and let $\lambda_1 = 1$, (5) implies

$$\|A_n x - P_1 x\| \leq \|x\| \frac{M}{n} \sum_{j=2}^{m} \frac{2}{|1 - \lambda_j|},$$

where $M = \max_{2 \leq j \leq m} \|P_j\|$. Thus $\{A_n\}$ converges uniformly to P_1. If $1 \notin \sigma(A)$, it is obvious that $\{A_n\}$ converges uniformly to zero. Therefore $(d) \Rightarrow (a)$ holds.

$(d) \Rightarrow (e)$. It is shown in a similar way.

$(a) \Rightarrow (c)$. See [1](Corollary 3, p.662). □

Theorem 8 *Let A be a bounded linear operator on E such that each spectral point of A on the unit circle is a pole and $\left\{\frac{A^n}{n}\right\}$ converges weakly to zero. Then*

(a) A is uniformly ergodic.

(b) In the case of $E = L^p$ with $1 \leq p \leq \infty$, A is p.e. as well.

Proof Since $\left\{\frac{A^n}{n}\right\}$ converges weakly to zero, we infer, by lemma 6 in ([1],p.709), that the spectral radius of A is not greater than 1. An application of Lemma 6 above shows that A has only finite number of spectral points on the unit circle, and all such spectral points are simple poles. The desired conclusions follow from Lemma 7 and Theorem 5 above. □

Let $A \in \mathcal{B}(E)$. If there is a compact operator Q in $\mathcal{B}(E)$ and a positive integer m so that $\|A^m - Q\| < 1$, A is called to be quasi-compact. It is a useful type of operators. Each of compact operators is, of course, quasi-compact.

Corollary Let A be a quasi-compact operator on E such that $\left\{\frac{A^n}{n}\right\}$ converges weakly to zero, then A is u.e.. In the case of $E = L^p$ with $1 \leq p \leq \infty$, A is p.e. as well. In particular, if A is a compact contraction, the same conclusions hold.

Proof By Theorem 3 in ([1], p.711), we see that A satisfies the hypothesis of Theorem 8 above and hence all the desired results hold. □

Remark The first assertion of this Corollary is nothing but the Yosida-Kakutani theorem on uniform ergodicity of quasi-compact operators ([6]; [1], p.711),which is a special case of Theorem 8,(a). In essense, it is in the Yosida-Kakutani theorem asked for A that each eigenspace of A corresponding to an eigenvalue on the unit circle is of finite dimension, while in Theorem 8 above that restriction is dispensed with.

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摘　要

本文证明了关于$L^p(1 \leq p \leq \infty)$空间中线性算子与非线性算子的几个点态遍历定理，还推广了Yosida与Kakutani关于拟紧算子的一致遍历定理.