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Differential Subordination and Conformal Mappings *

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Abstrict In this paper we give a sufficient condition on $D^{\delta}g(z)$ such that with the condition. $\mathrm{Re}\left\{(1-\alpha)\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}+\alpha\frac{D^{\delta+2}f(z)}{D^{\delta+2}g(z)}\right\}>\rho$, $\rho<1$, it gnarantees that in the unit disk E implies $\mathrm{Re}\left\{\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}\right\}>\rho^0>\rho$, $z\in E$. Some interesting applications of this result are also given.

Key words Hadamard product, differential superdination.

1. Introduction

Let V denote the class of functions $f(z)=\sum\limits_{n=2}^{\infty}a_nz^n$ analytic in the unit disc E. Let $D^{\delta}f(z)=(z/(1-z)^{\delta+1})\star f(z)$, where $\delta\geq-1$, $f\in V$ and $\star$ denotes the Hadamard product of two analytic functions in E [a].

Very recently, Ponnusamy and Jureja [7] have proved that if $g\in V$ satisfies

$$\mathrm{Re}\left\{\frac{D^{\delta+2}f(z)}{D^{\delta+1}g(z)}\right\}>0,\quad z\in E,\quad \delta\geq-1,$$

then for $f\in V,\rho<1$ and $\alpha\geq0$

$$\mathrm{Re}\left\{(1-\alpha)\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}+\alpha\frac{D^{\delta+2}f(z)}{D^{\delta+2}g(z)}\right\}>\rho,\quad z\in E$$

implies $\mathrm{Re}\left\{\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}\right\}>\rho,\quad z\in E.$

The case $\delta=-1$ of the above result gives.

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Theorem 1.1 *If $g \in V$ is starlike in E, $\rho < 1$ and $\alpha \geq 0$, then for $f \in V$*

$$\mathrm{Re}\left\{(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{f'(z)}{g'(z)}\right\} > \rho, \quad \text{in } E$$

implies $\mathrm{Re}\left\{\frac{f(z)}{g(z)}\right\} > \rho$, *in* E.

But in [1], Bulboaca has obtained the following result.

Theorem 1.2 *If $\rho < 1, \alpha$ a complex number with $\mathrm{Re}(\alpha) \geq 0$, $f, g \in V$ and $\mathrm{Re}(\alpha g(z)/zg'(z))$*

> 0, $z \in E$, then

$$\mathrm{Re}\left\{(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{f'(z)}{g'(z)}\right\} > \rho, \quad z \in E$$

implies $\mathrm{Re}\left\{\frac{f(z)}{g(z)}\right\} > \frac{M+2\rho}{M+2}$, $z \in E$, *where* $M = \mathit{inf}\{\mathrm{Re}(\alpha g(z)/zg'(z)) : z \in E\}$.

These two results suggest that there may exist some conditions on $D^{\delta}g(z)$ so that

$$\mathrm{Re}\left\{(1-\alpha)\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)} + \alpha\frac{D^{\delta+2}f(z)}{D^{\delta+2}g(z)}\right\} > \rho, \quad z \in E$$

implies $\mathrm{Re}\left\{\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}\right\} > \rho' > \rho$, $z \in E$.

In this section our aim is to find at least one such condition and to give some interesting applications of this result.

The basic tool in proving our result is the following lemma.

Lemma 1.1 *Let Ω be a set in the complex plane C and b be a complex number with the* $\mathrm{Re}(b) > 0$. *Suppose that the function $\psi : C^2xE \to C$ satisfies the condition $\psi(ix_2, y_1; z) \notin \Omega$ for all real $x_2, y_1 \leq -\frac{1}{2\mathrm{Re}(b)}|b - ix_2|^2$, $z \in E$. If $p(z) = b + p_1z + \cdots$ is analytic in E and $\psi(p(z), zp'(z); z) \subset \Omega$, then $\mathrm{Re}(p(z)) > 0$ in E.*

A more general form of this Lemma may be found in [4] and [5].

2. Main results

Theorem 2.1 *Let α be a complex number satisfying $\mathrm{Re}(\alpha) > 0$ and $\rho < 1$. Let $\delta \geq -1, f, g \in V$ and*

$$\mathrm{Re}\left\{\frac{D^{\delta+1}f(z)}{D^{\delta+2}g(z)}\right\} > \gamma, \quad 0 \leq \gamma < \mathrm{Re}(\alpha). \tag{2.1}$$

Then

$$\mathrm{Re}\left\{\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}\right\} > \frac{2\rho(\delta+2)+\gamma}{2(\delta+2)+\gamma}, \quad z \in E$$

whenever

$$\mathrm{Re}\left\{(1-\alpha)\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}+\alpha\frac{D^{\delta+2}f(z)}{D^{\delta+2}g(z)}\right\}>\rho,\quad z\in E. \tag{2.2}$$

Proof Let $\tau=(2\rho(\delta+2)+\gamma)/(2(\delta+2)+\gamma)$ and consider

$$p(z)=(1-\tau)^{-1}\left\{\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}-\tau\right\}. \tag{2.3}$$

This $p(z)$ is analytic in E and $p(0)=1$. We set $u(z)=\alpha\dfrac{D^{\delta+1}g(z)}{D^{\delta+2}g(z)}$ and observe that by (2.1) $\mathrm{Re}(u(z))>\gamma$. Since, for $\delta>-1$,

$$D^{\delta}f(z)=z+\sum_{k=2}^{\infty}\frac{\Gamma(k+\delta)}{(k-1)!\Gamma(1+\delta)}a_kz^k,$$

it can be easily verified that

$$z(D^{\delta}f(z))'=(\delta+1)D^{\delta+1}f(z)-\delta D^{\delta}f(z). \tag{2.4}$$

Using (2.4), (2.3) can be written as

$$(1-\alpha)\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}+\alpha\frac{D^{\delta+2}f(z)}{D^{\delta+2}g(z)} \tag{2.5}$$

$$=\tau+(1-\tau)\left[p(z)+\frac{u(z)}{\delta+2}zp'(z)\right]=\psi(p(z),zp'(z);z)$$

where $\psi(x,y;z)=\tau+(1-\tau)\left[x+\dfrac{u(z)}{\delta+2}y\right]$.

From (2.2) and (2.5) we obtain that

$$\{\psi(p(z),zp'(z);z):|z|<1\}\subset\Omega=\{w\in C:\mathrm{Re}(w(z))>\rho\}.$$

Now for all real $x_2,y_1\le -2^{-1}(1+x_2^2)$ we have

$$\mathrm{Re}\{\psi(ix_2,y_1;z)\}=\tau+\frac{(1-\tau)y_1}{\delta+2}\mathrm{Re}(u(z))\le\tau-\frac{(1-\tau)\gamma}{2(\delta+2)}\equiv\rho.$$

Hence $\psi(ix_2,y_1;z)\notin\Omega$. Thus by Lemma 1.1, $\mathrm{Re}(p(z))>0$ in E and hence $\mathrm{Re}\left\{\dfrac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}\right\}$ $>\tau$ in E. This proves our Theorem.

Remark For $\delta=-1$, Theorem 2.1 agrees with Theorem 3 of Bulboaca [1].

Corollary 2.1 *Let α be a real number with $\alpha\ge 1$ and $\rho<1$. Let $\delta\ge -1, f,g\in V$ and*

$$\left[\alpha\mathrm{Re}\left\{\frac{D^{\delta+1}g(z)}{D^{\delta+2}g(z)}\right\}\right]>\gamma,\quad 0\le\gamma<\alpha.$$

Then

$$\operatorname{Re}\left\{\frac{D^{\delta+2}f(z)}{D^{\delta+2}g(z)}\right\} > \frac{\alpha(2\rho(\delta+2)+\gamma)-(1-\rho)\gamma}{\alpha(2(\delta+2)+\gamma)}, \quad z \in E$$

whenever $\operatorname{Re}\left\{(1-\alpha)\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)} + \alpha\frac{D^{\delta+2}f(z)}{D^{\delta+2}g(z)}\right\} > \rho, \quad z \in E.$

Proof Proof follows from Theorem 2.1 (Since $\alpha \geq 1$).

Corollary 2.2 *Let* $\delta \geq -1$, $f, g \in V$ *and* $\operatorname{Re}\{D^{\delta+1}g(z)/D^{\delta+2}g(z)\} > \gamma$, $0 \leq \gamma < 1$, *then for* $\rho < 1$.

$$\operatorname{Re}\left\{\frac{D^{\delta+2}f(z)}{D^{\delta+2}g(z)}\right\} > \rho, \quad z \in E$$

implies $\operatorname{Re}\left\{\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}\right\} > \frac{2\rho(\delta+2)+\gamma}{2(\delta+2)+\gamma}, \quad z \in E.$

Proof Take $\alpha = 1$ in Theorem 2.1.

If we set

$$v(z) = \left\{\frac{D^{\delta+2}f(z)}{D^{\delta+2}g(z)}\right\} + \left[\frac{1}{\alpha} - 1\right]\left\{\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}\right\}$$

then for real $\alpha > 0$ and $\rho = 0$, Theorem 2.1 reduces to

(2.6) $\quad \operatorname{Re}(v(z)) > 0, \quad z \in E$ implies $\operatorname{Re}\left\{\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}\right\} > \frac{\gamma\alpha}{2(\delta+2)+\gamma\alpha}, \quad z \in E$

whenever $\operatorname{Re}\left\{D^{\delta+2}f(z)/D^{\delta+2}g(z)\right\} > \gamma$, $0 \leq \gamma < 1$. Let $\alpha \to \infty$. Then (2.6) is equivalent to

$$\left\{\frac{D^{\delta+2}f(z)}{D^{\delta+2}g(z)} - \frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}\right\} > 0 \text{ in } E \text{ implies } \operatorname{Re}\left\{\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}\right\} > 1 \text{ in } E$$

whenever $\operatorname{Re}\left\{D^{\delta+1}g(z)/D^{\delta+2}g(z)\right\} > \gamma$, $0 \leq \gamma < 1$.

In the following Theorem we shall extend the above result.

Theorem 2.2 *Let* $\delta \geq -1, \rho < 1, f, g \in V$ *and* $\operatorname{Re}\left\{D^{\delta+1}g(z)/D^{\delta+1}g(z)\right\} > \gamma$, $0 \leq \gamma < 1$. *If*

$$\operatorname{Re}\left\{\frac{D^{\delta+2}f(z)}{D^{\delta+2}g(z)} - \frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}\right\} > -\frac{(1-\rho)\gamma}{2(\delta+2)}, \quad z \in E,$$

then

$$\operatorname{Re}\left\{\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}\right\} > \rho, \quad z \in E$$

and

$$\operatorname{Re}\left\{\frac{D^{\delta+2}f(z)}{D^{\delta+2}g(z)}\right\} > \frac{\rho(2(\delta+2)+\gamma)-\gamma)}{2(\delta+2)} \text{ in } E$$

Proof It can be proved in a manner similar to that of Theorem 2.1.

Using Theorem 2.1 and Theorem 2.2 we can generalise and improve several other intresting results available in the literature by taking $g(z) \equiv z$. We will illustrate this as follows.

Theorem 2.3 *Let $\delta \geq -1, f \in V$ and $\rho < 1$. Then*

(a) for α a complex number satifying $\mathrm{Re}(\alpha) > 0$, we have

$$\mathrm{Re}\left\{(1-\alpha)\frac{D^{\delta+1}f(z)}{z} + \frac{D^{\delta+2}f(z)}{z}\right\} > \rho, \quad z \in E$$

implies $\mathrm{Re}\left\{\frac{D^{\delta+1}f(z)}{z}\right\} > \frac{2\rho(\delta+2)+\mathrm{Re}(\alpha)}{2(\delta+2)+\mathrm{Re}(\alpha)}, \quad z \in E.$

(b) For α real and $\alpha \geq 1$, we have

$$\mathrm{Re}\left\{(1-\alpha)\frac{D^{\delta+1}f(z)}{z} + \alpha\frac{D^{\delta+2}f(z)}{z}\right\} > \rho \quad \text{in } E$$

implies $\mathrm{Re}\left\{\frac{D^{\delta+1}f(z)}{z}\right\} > \frac{\rho(2\delta+5)+\alpha-1}{2(\delta+2)+\alpha}$ *in E.*

(c)

$$\mathrm{Re}\left\{\frac{D^{\delta+2}f(z)}{z} - \frac{D^{\delta+1}f(z)}{z}\right\} > -\frac{(1-\rho)}{2(\delta+2)} \quad \text{in } E$$

implies $\mathrm{Re}\left\{\frac{D^{\delta+2}f(z)}{z}\right\} > \frac{\rho(2\delta+5)-1}{2(\delta+2)}$ *in E.*

For $\delta = -1$, Theorem 2.3 follows.

Corollary 2.3 *Let $f \in V$ and $\rho < 1$. Then*

(a′) for α a complex number satisfying $\mathrm{Re}(\alpha) > 0$, we have

$$\mathrm{Re}\left\{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right\} > \rho \text{ in } E$$

implies $\mathrm{Re}\left\{\frac{f(z)}{z}\right\} > \frac{2\rho+\mathrm{Re}(\alpha)}{2+\mathrm{Re}(\alpha)}$ *in E,*

(b′) for α real and $\alpha \geq 1$, we have

$$\mathrm{Re}\left\{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right\} > \rho \text{ in } E$$

implies $\mathrm{Re}(f'(z)) > \frac{3\rho+\alpha-1}{2+\alpha}$ *in E,*

(c′)

$$\mathrm{Re}\left\{f'(z) - \frac{f(z)}{z}\right\} > -\frac{(1-\rho)}{2} \text{ in } E$$

implies $\mathrm{Re}(f'(z)) > \frac{3\rho-1}{2}$ *in E.*

For $\delta = 0$, Theorem 2.3 reduces to

Corollary 2.4 *Let $f \in V$ and $\rho < 1$. Then*

(a″) for α a complex number satisfying $Re(\alpha) > 0$, we have

$$\mathrm{Re}\left[f'(z) + \alpha z f''(z)\right] > \rho \text{ in } E$$

implies $\mathrm{Re}\left[f'(z)\right] > \dfrac{2\rho + \mathrm{Re}(\alpha)}{2 + \mathrm{Re}(\alpha)}$ *in* E,

(b″) for α real and $\alpha \geq 1/2$, we have

$$\mathrm{Re}\left[f'(z) + \alpha z f''(z)\right] > \rho \text{ in } E$$

implies $\mathrm{Re}\left[f'(z) + (zf'(z))'\right] > \dfrac{5\rho + 2\alpha - 1}{2 + \alpha}$ *in* E,

(c″) $$\mathrm{Re}\left[f''(z)\right] > -\frac{(1-\rho)}{2} \text{ in } E$$

implies $\mathrm{Re}\left[f'(z) + (zf'(z))'\right] > \dfrac{5\rho - 1}{2}$ *in* E.

Remark The cases (a′), (a″) and (b′) were obtained by Bulboaca [1] and also by Juneja and Ponnusamy [3]. The case (c′) was obtained by Ponnusamy and Karunakaran [6]. The cases (a′), (a″) and (b′) not only generalise results of Chichra [2] but also improve upon them.

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