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Banach 空间的保持端点

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摘 要

在本文中，我们定义了(R1),(R2),(R3) 和wMLUR, 它们是一些与严格凸较接近且所有端点都是保持端点的凸性. 我们还讨论了它们的性质以及它们之间的相互关系.

关键词: 保持端点.

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The Preserved Extreme Points in Banach Spaces *

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Abstract In this paper, we define the (R1),(R2),(R3) and wMLUR, which are some convexity similar to strict convexity and preserving all extreme points.

Key words Preserved extreme point, wMLUR, (R1),(R2),(R3)

Let X be a Banach space. The unit sphere of X is denoted by $S(X)$ and the unit ball of X is denoted by $U(X)$. By the natural embedding $x \to Jx, X$ may be considered as a subspace of X^{**}. An extreme point of the unit ball of Banach space is said to be a preserved extreme point, if it is an extreme point in the unit ball of the second dual. Peter Morris [1] showed that if X is a separable Banach space containing an isomorphic copy of C_0, then X is isomorphic to a strictly convex space E such that no extreme point of E is preserved. In this paper, we discuss some convexities which are similar to strictly convesity and preserve all the extreme points.

Definition 1 *If* $\|x-(x_n+y_n)/2\| \to 0$ *implies* $x_n - y_n \xrightarrow{w} 0$, *where* $x, x_n, y_n \in S(X)$, *then we say that* X *is* wMLUR.

Definition 2 *If every* $x \in S(X), x$ *is an extreme point of* $U(X^{**})$, *i.e., every point of* $S(X)$ *is a preserved extreme point, then we say that* X *is* (R1).

Definition 3 *If* $\|x + x^{**}\| = 2$ *implies* $x = x^{**}$, *where* $x \in S(X), x^{**} \in S(X^{**})$, *we say that* X *is* (R2).

Definition 4 *If* $(x_n+y_n)/2 \xrightarrow{w} x$ *implies* $x_n \xrightarrow{w} x$ *and* $y_n \xrightarrow{w} x$, *where* $x_n, y_n \in S(X)$, *then we say that* X *is* (R3).

It is easy to see that (R2) implies (R1), and if a Banach space X is (R3), then X is wMLUR.

Theorem 5 *Let* X *be a Banach space and* X^* *is smooth, then the following statements hold:*

(1) X *is* (R1);

(2) X *is* (R3).

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Proof (1) Let $x \in S(X), x^{**}, y^{**} \in S(X^{**})$, with $x = (x^{**} + y^{**})/2$, then there exist $f_x \in S(X^*)$, such that $f_x(x) = 1$. Since $x^{**}, y^{**} \in S(X^{**})$, we know that $x^{**}(f_x) = 1$ and $y^{**}(f_x) = 1$. As X^* is smooth, so $x = x^{**} = y^{**}$, hence X is (R1).

(2) Suppose that $(x_n + y_n)/2 \xrightarrow{w} x$, since X^* is smooth, from [2], we know that $U(X^{**})$ is w^*-sequentially compact, so $x_{n_k} \xrightarrow{w^*} x^{**}, y_{n_k} \xrightarrow{w^*} y^{**}$. As $(x_n + y_n)/2 \xrightarrow{w^*} x$ in X^{**}, so $x = (x^{**} + y^{**})/2$, by (1), we know that $x = x^{**} = y^{**}$, thus $x_{n_k} \xrightarrow{w^*} x, y_{n_k} \xrightarrow{w^*} x$ in X^{**}, hence $x_n \xrightarrow{w} x$ and $y_n \xrightarrow{w} x$ in X (Otherwise, we have $f \in S(X^*)$ and $\epsilon_0 > 0$ and $\{x_{n_k}\} \subset \{x_n\}$ such that $|f(x_{n_k}) - f(x)| \geq \epsilon_0$. By above discuss, we have $\{x_{n_{k_1}}\} \subset \{x_{n_k}\}$ such that $x_{n_{k_1}} \xrightarrow{w} x$ in X^{**}, that contradicts to $|f(x_{n_k}) - f(x)| \geq \epsilon_0$).

Theorem 6 *If X^* is smooth and X is very smooth, then Banach space X is* (R2).

Proof Let $x \in S(X), x^{**} \in S(X^{**})$ with $\|x + x^{**}\| = 2$, then we have $f_n \in S(X^*)$ such that $(x + x^{**})(f_n) \to 2$, so $x^{**}(f_n) \to 1$ and $f_n(x) \to 1$, since X is very smooth, the latter implies that $f_n \xrightarrow{w} f_x$, where $f_x \in S(X^*), f_x(x) = 1$, thus $f_x(x) = x^{**}(f_x) = 1$. As X^* is smooth, so $x = x^{**}$, hence X is (R2).

It is easy to see the following theorem holds.

Theorem 7 *If X^{**} is strictly convex, then Banach space X is* (R2).

Theorem 8 *If X^* is* (R2), *then X is very smooth.*

Proof If $x \in S(X), f_x \in A(x)$ and $F_x \in S(X^{***})$ with $f_x(x) = 1 = F_x(x)$, then $\|f_x + F_x\| = 2$. Since X^* is (R2), we have $f_x = F_x$, hence X is very smooth.

Corollary 9 *X^{**} is* (R2) *if and only if X is reflexive and X is strictly convex.*

Theorem 10 *If Banach space X is reflexive, then the following statements are equivalent:*

(1) *X is strictly convex;*
(2) *X is* (R1);
(3) *X is* (R2);
(4) *X is* (R3);
(5) *X is* wMLUR.

Proof (1) $\Rightarrow$ (5). If X is strictly convex and $z, x_n, y_n \in S(X)$ with $\|z - (x_n + y_n)/2\| \to 0$, then since X is reflexive, we have $x_{n_k} \xrightarrow{w} x$ and $y_{n_k} \xrightarrow{w} y$, so $(x + y)/2 = z$. As X is strictly convex, so $x = y = z$, thus $x_n \xrightarrow{w} x$ and $y_n \xrightarrow{w} x$, hence X is wMLUR.

(5)$\Rightarrow$ (1) If X is wMLUR and $x, y, z \in S(X)$ with $x = (y + z)/2$, then let $x_{2k} = y, x_{2k+1} = z$ and $y_{2k} = z, y_{2k+1} = y$, we have $\|z - (x_n + y_n)/2\| = 0$, so $x_n \xrightarrow{w} x$, thus $x = y = z$, hence X is strictly convex.

By (1)$\Leftrightarrow$ (5) and theorem 5, we can easy to see that (5)$\Rightarrow$ (1)$\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5).

Theorem 11 *If Banach space X is* (R1), *and X^* is separable, then X is* wMLUR.

Proof Let $x, x_n, y_n \in S(X)$ with $\|x - (x_n + y_n)/2\| \xrightarrow{w} 0$, since X^* is separable, we know that $U(X^{**})$ is w^*-sequentially compact, so we have $x_{n_k} \xrightarrow{w^*} x, y_{n_k} \xrightarrow{w^*} y$ in X^{**}. As

$(x_n+y_n)/2 \xrightarrow{w} x$ in X, so $(x_n+y_n)/2 \xrightarrow{w^*} x$ in X^{**}, thus $x=(x^{**}+y^{**})/2$. Since X is (R1), we have $x=x^{**}=y^{**}$, so $x_{n_k} \xrightarrow{w} x, y_{n_k} \xrightarrow{w} x$ in X, hence $x_n \xrightarrow{w} x, y_n \xrightarrow{w} x$, i.e., X is (R3).

Theorem 12 *If separable Banach space X contains no subspace isomorphic to l^1, then the following statements are equivalent:*

(1) X *is* (R1);
(2) X *is* (R2);
(3) X *is* (R3);
(4) X *is* wMLUR.

Proof (3)⇒ (4). Easy.

(4)⇒ (2). Let $x \in S(X), x^{**} \in S(X^{**})$ with $\|x+x^{**}\|=2$, since $l^1 \not\hookrightarrow X$, we have $x_n \in S(X)$ such that $x_n \xrightarrow{w^*} x^{**}$. For any $\epsilon > 0$, there is an $f_\epsilon \in S(X^*)$ such that $x(f_\epsilon)+x^{**}(f_\epsilon) > 2-\epsilon/2$, for fixed f_ϵ, since $x_n \xrightarrow{w^*} x^{**}$, there exist N such that whenever $n > N, x_n(f_\epsilon) > x^{**}(f_\epsilon)-\epsilon/2$, so $x(f_\epsilon)+x_n(f_\epsilon)) > 2-\epsilon$, thus $\|x+x_n\| \geq 2-\epsilon$. i.e., $\|x+x_n\| \to 2$. As X is wMLUR, we have $x_n \xrightarrow{w} x$, so $x_n \xrightarrow{w^*} x$ in X^{**}, hence $x=x^{**}$. i.e. X is (R2).

(2)⇒ (1). Easy.

(1)⇒ (3). Let $x, x_n, y_n \in S(X)$ with $(x_n+y_n)/2 \xrightarrow{w} x$, since $l^1 \not\hookrightarrow X$, we have $x_{n_k} \xrightarrow{w^*} x^{**}, y_{n_k} \xrightarrow{w^*} y^{**}$. As $(x_n+y_n)/2 \xrightarrow{w} x$, so $(x_n+y_n)/2 \xrightarrow{w^*} x$ in X^{**}, thus $(x^{**}+y^{**})/2=x$, since X is (R1). we have $x=x^{**}=y^{**}$, so $x_{n_k} \xrightarrow{w^*} x, y_{n_k} \xrightarrow{w^*} x$ in X^{**}, hence $x_n \xrightarrow{w} x$ and $y_n \xrightarrow{w} x$ in X.

Finally, it is easy to see the following theorem holds.

Theorem 13 *If Banach space X contains no subspace isomorphic to l^1, and X has* (H), *then the following statements are equivalent:*

(1) X is strictly convex.
(2) X *is* wMLUR.
(3) X *is* MLUR.
(4) X *is* (R1).
(5) X *is* (R2).
(6) X *is* (R3).

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