This is a contradiction. Hence $x \notin \overline{H-F}$, and $H-F \in 2^x$.

Now, we prove that $H-F$ is a cluster point of the net $\{A_\alpha, \alpha \in M\}$. Otherwise, then there is a neighborhood $\langle U_1, \cdots, U_l\rangle$ of $H-F$, and there is $\alpha_0 \in M$ such that $A_\alpha \notin \langle U_1, \cdots, U_l\rangle$ for any $\alpha \in M$ if $\alpha_0 \le \alpha$. Without loss of generality, suppose $F \neq \emptyset$ and $F \bigcap (\bigcup_{i=1}^{l} U_i) = \emptyset$. Then $A_\alpha \not\subset \bigcup_{i=1}^{l} U_i$ or there exists $j \in \{1, \cdots, l\}$ such that $A_\alpha \cap U_j = \emptyset$ for $\alpha_0 \le \alpha$. Set $V = X - \overline{\bigcup_{\alpha\in M} A_\alpha}$, since $F \bigcap \overline{\bigcup_{\alpha\in M} A_\alpha} = \emptyset$, $H \in \langle U_1, \cdots, U_l, V\rangle$, there is $\beta \in M$ and $\alpha_0 \le \beta$ such that $A_\beta \cup F \in \langle U_1, \cdots, U_l, V\rangle$. Hence $A_\beta \subset \bigcup_{i=1}^{l} U_i$ and $A_\beta \cap U_j \neq \emptyset$ for every $j \le l$: a contradiction. Therefore $H-F$ is a cluster point of the net $\{A_\alpha, \alpha \in M\}$, and it is a cluster point of $\{A_\alpha, \alpha \in D\}$. Hence, 2^X is m-compact.

It is easy to see that countable compactness of X does not imply countable compactness of 2^X.

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超空间2^X的局部覆盖性质

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摘 要

本文讨论了超空间2^X 的某些局部覆盖性质，并给出下面二个结果:

定理1 设X 是T_2 空间，则2^X 紧当且仅当2^X 是局部meta-Lindelöf 空间.

定理2[1] 设X 是T_1 空间，则2^X m- 紧当且仅当2^X 是局部m- 紧.

关键词: 超空间，m- 紧性.

[1] 本文定稿后，作者看到董笑咏，王世坤一文在X 是正则空间的条件下给出$2^X m$- 紧当且仅当2^X 局部m- 紧的结果，且证明用到选择公理，其方法与本文定理2 完全不同(见自然科学论文选，内蒙民族师院1988).

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Local Covering Properties of Hyperspaces 2^X *

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Abstrict We discuss some local covering properties of 2^X, and prove that some covering properties are equivalent to local properties in 2^X.

Key words Hyperspace, meta-Lindelöf, m-compactness.

Covering properties of hyperspaces have been widely discussed, and some important results were obtained. For example, J. Keesling [2] gave:

Theorem *The folloming are equivalent:*

(a) X *is compact.*
(b) 2^X *is compact.*
(c) 2^X *is Lindelöf.*
(d) 2^X *is paracompact.*
(e) 2^X *is metacompact.*
(f) 2^X *is meta-Lindelöf.*

In this paper we consider Local covering properties.

Let X be a topological space. 2^X the space of closed subsets of X with Vietoris topology which we now refer to as the hyperspace of X, and

$$\langle E_1,\cdots,E_n\rangle=\{E\in 2^X: E\subset\sum_{i=1}^{n}E_i,\quad E\bigcap E_i\neq\emptyset \text{ for all } i=1,\cdots,n\},$$

here, $E_i\subset X$ for each $i\leq n$.

X is a meta-Lindelöf space, if each open cover has a point countable open refinement.

X is an m-compact space, if each open cover with cardinal is less than or equal to m has a finite subcover.

X is a locally meta-Lindelöf space, if there is neighborhood U of x such that $\overline{U}$ is a meta-Lindelöf space for each $x\in X$.

The definition of local m-compact space is obvious.

Lemma 1([2]) *If* N *denotes the set of natural numbers with discrete topology, then* 2^N *is not meta-Lindelöf.*

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Theorem 1 *Let X be a T_2 space. Then the following statements are equivalent:*

(1) *X is compact.*

(2) *2^X is compact.*

(3) *2^X is meta-Lindelöf.*

(4) *2^X is locally meta-Lindelöf.*

(5) *there is $\{E_1, \cdots, E_n\} \subset 2^X$ such that $E_i - \bigcup_{j \neq i} E_j \neq \emptyset$ for each $i \leq n$, $X \in \langle E_1, \cdots, E_n \rangle$ and $\langle E_1, \cdots, E_n \rangle$ is a meta- Lindelöf space.*

Proof (1)⇒ (2) can be obtained from Theorem 4.2 of [1]. It is obvious that (2)⇒ (3) and (3)⇒ (4). Now we prove that (4)⇒ (5):

Suppose that $X \in \langle \overline{U_1}, \cdots, \overline{U_n} \rangle$, and $\langle \overline{U_1}, \cdots, \overline{U_n} \rangle$ is a meta-Lindelöf space. Here $U_1, \cdots, U_n$ are open sets in X. If $\overline{U_1}, \cdots, \overline{U_n}$ do not satisfy (5), without loss of generality we suppose that $\overline{U_i} \neq \overline{U_j}$ for $i \neq j$, $\overline{U_1} \subset \bigcup_{i \neq 1} \overline{U_i}$ and $\overline{U_i} - \overline{U_1} \neq \emptyset$ for $i \neq 1$. Set $U_1^1 = U_1$, $U_i^1 = U_i - \overline{U_1}$ for $i \neq 1$. Then $U_i^1 \neq \emptyset$ and $U_1^1 \cap U_i^1 = \emptyset$ for $i \neq 1$. It is easy to see that $U_i^1 \subset U_i$ and $\langle \overline{U_1^1}, \cdots, \overline{U_n^1} \rangle \subset \langle \overline{U_1}, \cdots, \overline{U_n} \rangle$. So $\langle \overline{U_1^1}, \cdots, \overline{U_n^1} \rangle$ is a meta-Lindelöf space.

We assert that $X \in \langle \overline{U_1^1}, \cdots, \overline{U_n^1} \rangle$: for any $x \in X$, if $x \in \overline{U_1}$, then $x \in \overline{U_1^1}$, if $x \notin \overline{U_1}$, then there exists $i \leq n$ such that $x \in \overline{U_i}$. So, $x \in \overline{U_i^1}$.

If $\overline{U_i^1} - \bigcup_{j \neq i} \overline{U_j^1} \neq \emptyset$ for each $i \leq n$, the proof is completed. Otherwise, without loss of generality suppose $\overline{U_i^1} \neq \overline{U_j^1}$ for $i \neq j$ $(i, j \leq n)$, and $\overline{U_2^1} \subset \bigcup_{i \neq 2} \overline{U_i^1}$ and $\overline{U_i^1} - \overline{U_2^1} \neq \emptyset$ for $i \neq 2$. Set $U_2^2 = U_2^1$, $U_i^2 = U_i^1 - \overline{U_2^1}$ $(i \neq 2)$, then $U_1^2 = U_1^1 - \overline{U_2^1} = U_1^1$. Similarly, we can prove that $X \in \langle \overline{U_1^2}, \cdots, \overline{U_n^2} \rangle$ and $\langle \overline{U_1^2}, \cdots, \overline{U_n^2} \rangle$ is a meta-Lindelöf space, and $U_1^2 \cap U_i^2 = \emptyset$ for $i \neq 1$, and $U_2^2 \cap U_i^2 = \emptyset$ for $i \neq 2$.

Repeating above process, a collection consisting of open sets which satisfy (5) will be obtained in finite steps.

(5)⇒ (1): It can be completed in two steps.

a) First we prove that X is a countably compact space, and E_i is countably compact for each $i \leq n$.

Otherwise, there is a sequence $\{x_i\}$ with no cluster point in X. Let $N = \{x_i\}$. Without loss of generality, when $i \leq m \leq n$, suppose that $N \subset \bigcup_{i=1}^{m} E_i$ and $N \bigcap (\bigcup_{i=1}^{n-m} E_{m+i}) = \emptyset$. Choose $x_i^* \in E_{m+i} - \bigcup_{j \neq m+i} E_j$, and set $F = \{x_1^*, \cdots, x_{n-m}^*\}$, $\mathcal{F}_1 = 2^N$, $\mathcal{F}_2 = \{E \cup F : E \in 2^N\}$. Here, suppose $F \neq \emptyset$. Now, we define a mapping $f : \mathcal{F}_2 \to \mathcal{F}_1$ such that $f(E \cup F) = E$ for any $E \cup F \in \mathcal{F}_2$. Obviously, f is a bijection.

Suppose that $E \in \mathcal{F}_1$. Let $\langle U \rangle \cap \mathcal{F}_1$ be an open neighborhood of E. Then $E \cup F \in \langle U, V \rangle \cap \mathcal{F}_2$, here $V = (\bigcup_{i=1}^{n-m} E_{m+i})^0$. Thus, for any $H \in 2^N$, if $H \cup F \in \langle U, V \rangle$, then $H \subset U$. Hence $f(H \bigcup F) \in \langle U \rangle \bigcap \mathcal{F}_1$, and $f[\langle U, V \rangle \bigcap \mathcal{F}_2] \subset \langle U \rangle \bigcap \mathcal{F}_1$.

If $\langle X, U \rangle \cap \mathcal{F}_1$ is an open neighborhood of E, set $V = U - F$. Then $\langle X, V \rangle \cap \mathcal{F}_1$ and

$\langle X,V\rangle \cap \mathcal{F}_2$ is an open neighborhood of E and $E\cup F$ respectively. Obviously, we have

$$f[\langle X,V\rangle \bigcap \mathcal{F}_2] \subset \langle X,V\rangle \bigcap \mathcal{F}_1 \subset \langle X,U\rangle \bigcap \mathcal{F}_1,$$

and f is continuous.

We may prove that f^{-1} is also continuous. For any $E\cup F \in \mathcal{F}_2$, let $\langle U\rangle \cap \mathcal{F}_2$ be an open neighborhood of $E\cup F$. Then $E\cup F\subset U$. Set $V = U - F$, since $E\cap F = \emptyset$, $E \in \langle V\rangle \cap \mathcal{F}_1$. Thus $f^{-1}[\langle V\rangle \bigcap \mathcal{F}_1] \subset \langle U\rangle \bigcap \mathcal{F}_2$.

If $\langle X,U\rangle \cap \mathcal{F}_2$ is an open neighborhood of $E\cup F$, then $U\cap (E\cup F) \neq \emptyset$. If $U \cap F \neq \emptyset$, then $f^{-1}[\mathcal{F}_1] \subset \langle X,U\rangle \cap \mathcal{F}_2$. If $U\cap F = \emptyset$, then $U\cap E \neq \emptyset$. Hence $E \in \langle X,U\rangle \cap \mathcal{F}_1$, and

$$f^{-1}[\langle X,U\rangle \bigcap \mathcal{F}_1] \subset \langle X,U\rangle \bigcap \mathcal{F}_2.$$

So f^{-1} is continuous.

Thus $f : \mathcal{F}_2 \to \mathcal{F}_1$ is a homeomorphism.

Next, we prove that $\mathcal{F}_2$ is a closed subset of $\langle E_1,\cdots,E_n\rangle$. By 2.2 of [1], it is easy to see that $2^{N\cup F}$ is a closed subset of 2^X, and

$$2^{N\cup F} = 2^N \bigcup \{E \bigcup \{x_1^*\} : E \in 2^N\} \bigcup \cdots \bigcup \{E \bigcup F : E \in 2^N\} \bigcup 2^F.$$

Suppose $M \in 2^{N\cup F} - (\mathcal{F}_2 \cup 2^F)$, then $F - M \neq \emptyset$. Thus, $\langle X - (F - M)\rangle$, and $M \in \langle X - (F-M)\rangle \cap \mathcal{F}_2 = \emptyset$. Since $2^F \cap \mathcal{F}_2 = \emptyset$, $\mathcal{F}_2$ is a closed subset of $2^{N\cup F}$. Hence $\mathcal{F}_2$ is a closed subset of 2^X, and it is also closed in $\langle E_1,\cdots,E_n\rangle$.

Since $\langle E_1,\cdots,E_n\rangle$ is meta-Lindelöf, and 2^N is homeomrphic to $\mathcal{F}_2$, therefore 2^N is a meta-Lindelöf space. Note that N is a discrete closed subset of X. This is a contradiction to Lemma 1.

b) Now we prove that E_i is a meta-Lindelöf space for each $i \le n$.

Define a mapping

$$\psi : E_1 \times \cdots \times E_n \to \langle E_1,\cdots,E_n\rangle$$

such that $\psi(x) = \{x_1,\cdots,x_n\}$ for any $x = (x_1,\cdots,x_n) \in E_1\times\cdots\times E_n$. Then $\psi : E_1\times\cdots\times E_n \to \psi[E_1\times\cdots\times E_n]$ is a perfect mapping and $\psi[E_1\times\cdots\times E_n]$ is closed in $\langle E_1,\cdots,E_n\rangle$ (see the proof of Theorem 1 in [6]), and $\psi[E_1\times\cdots\times E_n]$ is meta-Lindelöf. So, $E_1\times\cdots\times E_n$ is a meta-Lindelöf space (see table II of [5]). It is easy to see that E_i is homeomorphic to a closed subset of $E_1\times\cdots\times E_n$, and so E_i is a meta-Lindelöf space for each $i \le n$.

By (a) and (b), E_i is a compact subset for each $i \le n$ (see [4]). So, $X = \bigcup_{i=1}^{n} E_i$ is a compact space.

By the above theorem we know that some covering properties and their Local properties in 2^X are equivalent.

The following lemma is clear.

Lemma 2 *Let m be an infinite cardinal. Then X is an m- compact space if and only if each net $\{x_\alpha, \alpha \in D\}$ has a cluster point in X whenever $|D| \le m$.*

Theorem 2 *Let X be a T_1 space. Then the following statements are equivalent.*

(1) 2^X *is m-compact,*
(2) 2^X *is locally m-compact,*
(3) There is $\{E_1,\cdots,E_n\}\subset 2^x$ *such that* $E_i-\bigcup_{j\neq i}E_j\neq\emptyset$ *for any* $i\leq n$, *and* $\langle E_1,\cdots,E_n\rangle$ *is an m-compact space and* $X\in\langle E_1,\cdots,E_n\rangle$.

Proof We only prove that (3)$\Rightarrow$ (1).

Let $\{A_\alpha,\alpha\in D\}$ be a net in 2^X and $|D|\leq m$. Since $X\in\langle E_1,\cdots,E_n\rangle$, there exists some E_i, without loss of generality suppose $i=1$, for any $\alpha\in D$ there is $\beta\in D$ such that $\alpha\leq\beta$ and $A_\beta\cap E_1\neq\emptyset$. Set $D_1=\{\alpha\in D: A_\alpha\cap E_1\neq\emptyset\}$, then $\{A_\alpha,\alpha\in D_1\}$ is a subnet of $\{A_\alpha,\alpha\in D_1\}$. Thus after a suitabe relabelling, there will be a subnet $\{A_\alpha,\alpha\in D_k\}$ of the net $\{A_\alpha,\alpha\in D\}$ such that $A_\alpha\cap E_i\neq\emptyset$ for any $\alpha\in D_k$ and $i\leq k$, and for any $i\geq k+1$ there is $\alpha_i\in D_k$ such that $A_\alpha\cap E_i=\emptyset$ whenever $\alpha_i\leq\alpha$ for any $\alpha\in D_k$ $(i=k+1,\cdots,n)$. Hence there exists $\beta\in D_k$ such that $\alpha_i\leq\beta$ for all $i=k+1,\cdots,n$. So $\{A_\alpha,\alpha\in D_k$ and $\beta\leq\alpha\}$ is a subnet of $\{A_\alpha,\alpha\in D_k\}$, and is also a subnet of $\{A_\alpha,\alpha\in D\}$. This subnet is denoted by $\{A_\alpha,\alpha\in M\}$. Obviously, $|M|\leq m$ and $A_\alpha\cap E_i=\emptyset$ for any $\alpha\in M$ and $i\geq k+1$.

Choose $x_i\in E_{k+i}-\bigcup_{j\neq k+i}E_j$, set $F=\{x_1,\cdots,x_{n-k}\}$. Then $\{A_\alpha\cup F,\alpha\in M\}$ is a net in $\langle E_1,\cdots,E_n\rangle$. Since $\langle E_1,\cdots,E_n\rangle$ is m-compact, $\{A_\alpha\cup F,\alpha\in M\}$ has a cluster point $H\in\langle E_1,\cdots,E_n\rangle$.

We assert that $H-F\in 2^x$.

First we prove that $H-F\neq\emptyset$. Otherwise, $H\subset F$. It is easy to see that $H\supset F$ (Otherwise, then $H\in\langle X-(F-H)\rangle$, and there is $A_\alpha\cup F\in\langle X-(F-H)\rangle$: a contradiction). Thus, $H=F$. Set $V=(\bigcup_{i=k+1}^{n}E_i)^o$. Then $H\in\langle V\rangle$. Thus, there is $\alpha\in M$ such that $A_\alpha\cup F\in\langle V\rangle$, and there exists E_{k+i} such that $A_\alpha\cap E_{k+i}\neq\emptyset$, a contradiction.

Next, we prove that $H-F$ is closed. Suppose $x\notin H-F$, then $x\notin H$ or $x\in H\cap F$. If $x\notin H$, then $(H-F)\cap(X-H)=\emptyset$. If $x\in H\cap F$, then there is $i\in\{1,\cdots,n-k\}$ such that $x\in E_{k+i}-\bigcup_{j\neq k+i}E_j=X-\bigcup_{j\neq k+i}E_j$. Assume that $x\in\overline{H-F}$, then

$$(X-\bigcup_{j\neq k+i}E_j)\bigcap(H-F)\neq\emptyset$$

and

$$(X-\bigcup_{j\neq k+i}E_j-\{x\})\bigcap(H-F)\neq\emptyset.$$

Hence $H\in\langle X,X-\bigcup_{j\neq k+i}E_j-\{x\}\rangle$, and there exists $\alpha\in M$ such that $A_\alpha\cup F\in\langle X,X-\bigcup_{j\neq k+i}E_j-\{x\}\rangle$. So, $(A_\alpha\cup F)\cap(X-\bigcup_{j\neq k+i}E_j-\{x\})\neq\emptyset$. Therefore we have

$$A_\alpha\bigcap E_{k+i}\supset A_\alpha\bigcap(X-\bigcup_{j\neq k+i}E_j-\{x\})\neq\emptyset.$$

This is a contradiction. Hence $x \notin \overline{H-F}$, and $H - F \in 2^x$.

Now, we prove that $H - F$ is a cluster point of the net $\{A_\alpha, \alpha \in M\}$. Otherwise, then there is a neighborhood $\langle U_1, \cdots, U_l\rangle$ of $H - F$, and there is $\alpha_0 \in M$ such that $A_\alpha \notin \langle U_1, \cdots, U_l\rangle$ for any $\alpha \in M$ if $\alpha_0 \leq \alpha$. Without loss of generality, suppose $F \neq \emptyset$ and $F \bigcap (\bigcup_{i=1}^{l} U_i) = \emptyset$. Then $A_\alpha \not\subset \bigcup_{i=1}^{l} U_i$ or there exists $j \in \{1, \cdots, l\}$ such that $A_\alpha \cap U_j = \emptyset$ for $\alpha_0 \leq \alpha$. Set $V = X - \overline{\bigcup_{\alpha \in M} A_\alpha}$, since $F \bigcap \overline{\bigcup_{\alpha\in M} A_\alpha} = \emptyset$, $H \in \langle U_1, \cdots, U_l, V\rangle$, there is $\beta \in M$ and $\alpha_0 \leq \beta$ such that $A_\beta \cup F \in \langle U_1, \cdots, U_l, V\rangle$. Hence $A_\beta \subset \bigcup_{i=1}^{l} U_i$ and $A_\beta \cap U_j \neq \emptyset$ for every $j \leq l$: a contradiction. Therefore $H - F$ is a cluster point of the net $\{A_\alpha, \alpha \in M\}$, and it is a cluster point of $\{A_\alpha, \alpha \in D\}$. Hence, 2^X is m-compact.

It is easy to see that countable compactness of X does not imply countable compactness of 2^X.

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摘 要

本文讨论了超空间2^X 的某些局部覆盖性质, 并给出下面二个结果:

定理1 设X 是T_2 空间, 则2^X 紧当且仅当2^X 是局部meta-Lindelöf 空间.

定理2[1] 设X 是T_1 空间, 则2^X m- 紧当且仅当2^X 是局部m- 紧.

关键词: 超空间, m- 紧性.

[1] 本文定稿后, 作者看到董笑咏, 王世坤一文在X 是正则空间的条件下给出$2^X m$- 紧当且仅当2^X 局部m- 紧的结果, 且证明用到选择公理, 其方法与本文定理2 完全不同(见自然科学论文选, 内蒙民族师院1988).