Journal of Mathematical Research & Exposition
Vol.14, No.4, Nov. 1994

Group Boolean Algebras *

W.B. Vasantha Kandasamy
(Dept. of Math., Indian Institute of Technology, India)

In this note we define a new algebraic structure called group Boolean algebra which are groups over Boolean algebras; analogous to group algebras which are groups over rings. We study only group Boolean algegras over the Boolean algebra $B = [0,1]$; as every other finite Boolean algebra is isomorphic to a direct product of $[0,1]$. Throughout this paper B denotes the Boolean algebra $B = [0,1]$ and G a group and BG the group Boolean algebra of group G over the Boolean algebra B.

Definition 1 *Let $B = [0,1]$ be the Boolean algebra of order 2 and G any group. The group Boolean algebra BG of G over B consists of all formal sums $\alpha = \sum \alpha_i g_i$ with $\alpha \in B$ and $g_i \in G$ such that* supp $\alpha = \{g_i/\alpha_i \neq 0\}$, *the support of α is finite; with the following operational rules.*

(i) $\sum \alpha_i g_i = \sum \beta_i g_i \Longleftrightarrow \alpha_i = \beta_i$ *for all* $g_i \in G$.

(ii) $\sum \alpha_i g_i + \sum \beta_i g_i = \sum(\alpha_i + \beta_i)g_i \quad \alpha_i, \beta_i \in B$ *and* $g_i \in G$.

(iii) $(\sum \alpha_i g_i)(\sum \beta_j g_j) = \sum \gamma_k g_k$ *where* $\gamma_k = \sum \alpha_i \beta_j$ *and* $g_i g_j = g_k$.

(iv) $1 \cdot g_i = g_i$ *for all* $g_i \in G$ *and* $1 \in B$. $1 \in G$ *and* $1 \in B$ *are identified to be* 1 *in* BG. *Since* $1 \in G$, B. $1 \subseteq BG$, *hence there is a natural embedding of* B *in* BG. *Thus* $b \to b1$ *is an embedding of* B *in* BG, *after identification of* $B \cdot l$ *with* B *we have* $B \subseteq BG$. *Clearly for all* $b \in B$ *and* $g \in G$ $bg = gb$. *Thus* $1 \in B$ *and* $1 \in G$ *is identified as the identity of* BG.

Example $B = [0,1]$ be the Boolean algebra of two elements and $G = \langle g | g^2 = 1 \rangle$ be the cyclic group of degree 2. $BG = \{0, 1, g, 1+g\}$.

Proposition 2 Let B be the Boolean algebra $[0,1]$ and G any group. The Group Boolean algebra BG is a semiring with 1.

Proof BG is obviously a semigroup under $+$. Further $1 \in BG$. Hence BG is a semiring with 1.

It is interesting and important to note that unlike a group algrbra, a group Boolean algebra is just a semiring.

*Received Aug. 22, 1992.

Theorem 3 *BG is a strict semiring.*

Proof As if $\alpha, \beta \in BG$. $\alpha + \beta = 0$ is possible if and only if $\alpha = 0$ and $\beta = 0$.

Theorem 4 *BG is an idempotent semigroup under addition.*

Proof As $1 + 1 = 1$ in BG we have BG to be an idempotent semigroup under $+$.

Theorem 5 *BG has no nontrivial zero divisors.*

Proof Since in B we have only 0 and 1 and $1 + 1 = 1$, we have $\alpha \cdot \beta = 0 \quad \alpha, \beta \in BG$. $(\alpha = \sum g_i \quad \beta = \sum h_j)$; $\alpha\beta = \sum g_i g_j = 0$ is possible if and only if $\alpha = 0$ or $\beta = 0$.

Theorem 6 *Let G be a group having an element of finite order and $B = [0,1]$. Then BG has no nontrivial idempotents with respect to multiplication.*

Proof Let $g \in G$ $(g \neq 11)$ such that $g^n = 1$. Clearly $(1+g+\cdots+g^{n-1})^2 = 1+g+\cdots+g^{n-1}$ as $1 + 1 = 1$ in BG. Hence the result.

Theorem 7 *Let G be a torsion free abelian group and $B = [0,1]$. The group ring BG has no nontrivial idempotents with respect to multiplication.*

Proof Let $\alpha = \sum g_i$ with $g_i \in G$. Since G is torsion free abelian; G is orderable. Let $g_1 < g_2 < \cdots < g_n$. Now from $\alpha^2 = \sum g_i^2 + \sum g_i g_j \quad (i \neq j)$, we have that g_1^2 is the smallest and g_n^2 is the largest element of supp α^2 which cannot equal any other element. Hence $\alpha^2 \neq \alpha$. Thus BG has no nontrivial idempotents.

Theorem 8 *Let G be a group having a finite subgroup H and $B = [0,1]$. Then BG has nontrivial idempotents with respect to multiplication.*

Proof Let $H = \{1, h_1, \cdots, h_n\}$ be the subgroup of G of finite order. Clearly if $\alpha = 1 + h_1 + \cdots + h_n$ then $\alpha^2 = \alpha$ using the fact $1 + 1 = 1$. Hence the theorem.

Theorem 9 *Let G be a torsion free non-abelian group and $B = [0,1]$ be the Boolean algebra. Then the group Boolean algebra BG has no nontrivial idempotents with respect to multiplication.*

Proof Suppose $\alpha^2 = \alpha$ where $\alpha = \sum g_i$ then $\{g_1, \cdots, g_n\}$ forms a finite subgroup of G, since $1 + 1 = 1$ in BG and BG is a strict semiring. These in turn imply that G has an element of finite order, a contradiction. Hence $\alpha^2 = \alpha$ is impossible in BG. Thus the semiring BG has no zero divisors.

Theorem 10 *BG is a non-commutative semiring if and only if G is a non-commutative group.*

Proof Obvious.

Theorem 11 *Let BG be the group Boolean ring of a finite group G over the Boolean algebra B. Then BG has nontrivial ideals.*

Proof Let $G = \langle 1, g_1, \cdots, g_n \rangle$. Take $\alpha = 1 + \sum g_i$. Then clearly $\alpha^2 = \alpha$. Now $\{0, 1 + g_1 +$

$\cdots + g_n\}$ is a nontrivial ideal of BG.

Problem If G is torsion free non-abelian or G is a infinite group which has no finite normal subgroups and $B = [0, 1]$. Can BG have nontrival ideals?

Reference

[1] Birkhoff, G., *Lattice Theory,* A.M.S. Providence, 1948.