Corollary[2] *Let R be a ring. Suppose that for all x, y in R, there exists an integer $n = n(x,y) > 1$ such that $x^n y = xy^n$. Then (i) N forms an ideal of R; (ii) $R = N \oplus R_1$, where R_1 is isomorphic to a subdirect sum of fields. In particular, if N is commutative, then R is commutative.*

Proof For all x, y in R, by hypothesis there exist integers $m = m(x,y) > 1$ and $n = n(x,y) > 1$ such that

$$x^n y = xy^n \text{ and } (x^n)^m y = x^n y^m.$$

Then $x^{mn} y = x^n y^m = x^n y y^{m-1} = xy^{m+n-1}$.

Since the equation $mn = m + n - 1$ has no integer solutions such that $m > 1$ and $n > 1$, there exist distinct integers $s = s(x,y) > 1$ and $t = t(x,y) > 1$ such that $x^s y = xy^t$. Then, the proof of corollary is now complete by Theorem.

References

[1] M. Hasanali and A. Yaqub, *Commutativity of rings with constraints on nilpotents and non-nilpotents,* Internat. J. Math. & Math Sci., **12(3)**(1989), 467—471.

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一类环的结构

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摘　要

设R是一结合环，N表示R的所有幂零元形成的集合. 本文证明了:

定理 如果环R满足条件: 对于任意$x_1, \cdots, x_k \in R$, 存在依赖于$x_1, \cdots, x_k$的字$\omega_1 = \omega_1(x_2, \cdots, x_k)$和$\omega_2 = \omega_2(x_1, \cdots, x_{k-1})$满足$|\omega_1|_{x_k} > 1$, $|\omega_2|_{x_1} > 1$, 和$|\omega_1| \neq |\omega_2|$, 使得$x_1\omega_1(x_2, \cdots, x_k) = \omega_2(x_1, \cdots, x_{k-1})x_k$, 这里$k > 1$是一固定整数，则(i)$N$形成$R$的理想; (ii)$R = N \oplus R_1$, 这里$R_1$同构于一些域的次直和. 特别地，如果$N$是交换的，则$R$也是交换的.

Journal of Mathematical Research & Exposition
Vol.14, No.4, Nov. 1994

The Structure of a Class of Rings *

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Abstract Let R be a ring, and N the set of all nilpotent elements of R. We prove the following

Theorem *If a ring R satisfies the following condition: for all $x_1, \cdots, x_k$ in R, there exist words $w_1 = w_1(x_2, \cdots, x_k)$ and $w_2 = w_2(x_1, \cdots, x_{k-1})$ depending on $x_1, \cdots, x_k$ such that $|w_1|_{x_k} > 1$, $|w_2|_{x_1} > 1$, and $|w_1| \neq |w_2|$. Suppose that*

$$x_1 w_1(x_2, \cdots, x_k) = w_2(x_1, \cdots, x_{k-1}) x_k,$$

where $k > 1$ is a fixed integer. Then
(1) N forms an ideal of R;
(2) $R = N + R_1$, where R_1 is isomorphic to a subdirect sum of fields. In particular, if N is commutative, then R is commutative.

Keywords Rings, subdirect sum, words.

Throughout, R will represent an associative ring, and N will denote the set of all nilpotent elements of R. Recently, Hasanali and Yaqub [1] proved that R is commutative if it satisfies the following three conditions : (i) N is commutative; (ii) $x^k y = xy^k$ for all $x, y \in R - N$; (iii) if $a \in N, b \in R$, and $k![a,b] = 0$, then $[a,b] = 0$, where $k > 1$ is a fixed integer.

We define a word $\omega(x_1, x_2, \cdots, x_k)$ in $x_1, x_2, \cdots, x_k$ to be a product in which each factor is x_i for some $i = 1, 2, \cdots, k$. By the x_i- length of word $\omega(x_1, x_2, \cdots, x_k)$, which we denote by $|\omega(x_1, \cdots, x_k)|x_i$, we shall mean the number of times x_i appears as factor in $\omega(x_1, x_2, \cdots, x_k)$; the sum $|\omega(x_1, \cdots, x_k)|x_1 + \cdots + |\omega(x_1, \cdots, x_k)|x_k$ will be called the length of $\omega(x_1, x_2, \cdots, x_k)$ and denote by $|\omega(x_1, \cdots, x_k)|$. We consider the following condition:

(*) For all $x_1, \cdots, x_k$ in R, there exist words $\omega_1 = \omega_1(x_2, \cdots, x_k)$ and $\omega_2 = \omega_2(x_1, \cdots, x_{k-1})$ depending on $x_1, \cdots, x_k$ such that $|\omega_1|x_k > 1, |\omega_2|x_1 > 1$, and $|\omega_1| \neq |\omega_2|$, and suppose

$$x_1 \omega_1(x_2, \cdots, x_k) = \omega_2(x_1, \cdots, x_{k-1}) x_k, \tag{1}$$

where $k > 1$ is a fixed integer.

Theorem *Let R be a ring satisfying condition (*). Then*

*Received Aug. 27, 1992.

(i) N forms an ideal of R;

(ii) $R = N \oplus R_1$, where R_1 is isomorphic to a subdirect sum of fields. In particular, if N is commutative, then R is commutative.

We begin with

Lemma *Let R be a semisimple ring satisfying condition (*). Then R is isomorphic to a subdirect sum of fields.*

Proof Let $x \in R$, and set $x_1 = x_2 = \cdots = x_k = x$ in (1), we get

$$x^m = x^n \tag{2}$$

for distinct integers $m = m(x) > 1$ and $n = n(x) > 1$. Without any loss of generality, we may assume that $m > n$.

If R is a division ring, then, by (2), $x = x^{m-n+1}$. Then R is a field.

Suppose now that R is a primitive ring. Note that condition $(*)$ is inherited by all subrings and all homomorphic images of R. Note also that no complete matrix ring $(D)_t$ over a division ring D $(t > 1)$ satisfies condition $(*)$, as may be illustrated by taking $x_1 = E_{12}$ and $x_2 = x_3 = \cdots = x_k = E$, we may assume that R is a division ring. Then R is a field.

If R is a semisimple ring, then R is isomorphic to a subdirect sum of primitive rings R_α each of which as a homomorphic image of R satisfies condition $(*)$, so each R_α is a field. Therefore, R is isomorphic to a subdirect sum of fields R_α.

We are now in a position to prove our theorem.

Proof of Theorem (i) Let J be the Jacobson radical of R. By the Lemma, R/J is isomorphic to a subdirect sum of fields F_α. Then $N \subseteq J$. On the other hand, for any x in J, by (2) we have $x^n = x^m = x^n \cdot x^{m-n}$, so $x^n = 0$, therefore $J \subseteq N$. Then $N = J$.

(ii) For each x in R, let $\overline{x}$ be the canonical image of x in R/J, and $\overline{x}_\alpha$ the image of $\overline{x}$ in R_α. Since R_α is a homomorphic image of R, by (2) we have $\overline{x_\alpha^m} = \overline{x_\alpha^n}$ so $\overline{x_\alpha} = \overline{x_\alpha^2}\ \overline{x_\alpha^{m-n-1}}$, and hence $\overline{x} = \overline{x^2}\ \overline{x^{m-n-1}}$, which gives

$$x - x^2 x^{m-n-1} \in J(= N).$$

Let $y = x^{m-n-1}$ and $z = x - x^n y^{n-1}$, then

$$\begin{aligned} z &= x - x^2y + x^2y - x^3y^2 + \cdots + x^{n-1}y^{n-2} - x^ny^{n-1} \\ &= (x - x^2y) + xy(x - x^2y) + \cdots + x^{n-2}y^{n-2}(x - x^2y) \in N, \\ x^ny^{n-1} &= x^{n+1}y^n = \cdots = x^{2n}y^{2n-1} = (x^ny^{n-1})^2y \quad (\text{ by } (2)), \end{aligned}$$

and

$$y(x^ny^{n-1}) = (x^ny^{n-1})y.$$

Let $R_1 = \{x \in R \mid \text{there exists } r_x \text{ in } R \text{ such that } x = x^2r_x \text{ and } x \cdot r_x = r_x \cdot x\}$. Then

$$x = z + x^ny^{n-1} \in N + R_1. \tag{3}$$

Since N is the set of nilpotent elements of R, we get $N \cap R_1 = 0$. It is clear that if R_1 is an ideal, then $R = N \oplus R_1$, and so $R_1 \simeq R/N$. Then it suffices to Prvoe that R_1 is an ideal of R.

Let $a \in R_1$, then $a = a^2 r_a$. Letting $e_a = ar_a$, we have

$$ae_a = a = e_a a \text{ and } e_a^2 = e_a.$$

If $a \in R_1$ and $u \in N$, then $au,\ ua \in N$. Let $x_1 = x_2 = \cdots = x_{k-1} = e_a$ and $x_k = au$ in (1), we have

$$au = \omega_2(e_a, \cdots, e_a)au = e_a\omega_1(e_a, \cdots, e_a, au) = (au)^{|\omega_1|x_k} e_a \quad \text{or } (au)^{|\omega_1|x_k}.$$

Since au is nilpotent and $|\omega_1|x_k > 1$, we obtain $au = 0$. A similar argument shows that $ua = 0$. Then $R_1 N = NR_1 = 0$.

For all a, b in R_1. Consider

$$(e_a e_b - e_a e_b e_a)^2 = 0 = (e_b e_a - e_a e_b e_a)^2. \tag{4}$$

Then, by (1) and (4), we have

$$e_a e_b - e_a e_b e_a = \omega_2(e_a, \cdots, e_a)(e_a e_b - e_a e_b e_a) = e_a \omega_1(e_a, \cdots, e_a, e_a e_b - e_a e_b e_a) = 0$$

and

$$e_b e_a - e_a e_b e_a = (e_b e_a - e_a e_b e_a)\omega_1(e_a, \cdots, e_a) = \omega_2(e_b e_a - e_a e_b e_a, e_a, \cdots, e_a)e_a = 0.$$

Hence $e_a e_b = e_a e_b e_a = e_b e_a$. Let $e = e_a + e_b - e_a e_b$. Then

$$e^2 = e, \quad ae = ea = a, \quad \text{and} \quad be = eb = b. \tag{5}$$

Let $x_1 = ab$, and $x_2 = \cdots = x_k = e$ in (1), by (5) we obtain

$$ab = ab\omega_1(e, \cdots, e) = \omega_2(ab, e, \cdots, e)e = (ab)^2 (ab)^{|\omega_2|_{x_1} - 2}.$$

Similarly, we get

$$a - b = (a - b)\omega_1(e, \cdots, e) = \omega_2((a - b), e, \cdots, e)e = (a - b)^2 (a - b)^{|\omega_2|_{x_1} - 2}.$$

Consider $(ab)(ab)^{|\omega_2|_{x_1}-2} = (ab)^{|\omega_2|_{x_1}-2}(ab)$ and $(a-b)(a-b)^{|\omega_2|_{x_1}-2} = (a-b)^{|\omega_2|_{x_1}-2}(a-b)$. Then $ab \in R_1$ and $a - b \in R_1$.

For $a \in R_1$ and $r \in R$, by (3) there exist r_1 in N and r_2 in R_1 such that $r = r_1 + r_2$. Then

$$\begin{aligned} ra &= (r_1 + r_2)a = r_1 a + r_2 a = r_2 a \in R_1, \\ ar &= a(r_1 + r_2) = ar_1 + ar_2 = ar_2 \in R_1. \end{aligned}$$

Hence R_1 is an ideal. This completes the proof of the theorem.

We conclude this note with the following

Corollary[2] *Let R be a ring. Suppose that for all x, y in R, there exists an integer $n = n(x,y) > 1$ such that $x^n y = xy^n$. Then (i) N forms an ideal of R; (ii) $R = N \oplus R_1$, where R_1 is isomorphic to a subdirect sum of fields. In particular, if N is commutative, then R is commutative.*

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