$\pi_1\epsilon_1 = 1_M$, where ϵ_1 is the natural injection from M to $M \oplus R$. This means that f is split. Thus M is projective.

Therefore every cyclic left R-module is projective. So R is Artinian semisimple, and we are done.

Acknowledgement *This research was supported by NSF of Gansu, China.*

References

[1] Faith, C., *Annihilator ideals, associated primes and Kasch-McCoy Commutative rings*, Comm. in Alg., **19(7)**(1991), 1867–1892.

[2] Huynh, D.Van and Smith, P.F., *Some rings characterised by their modules*, Comm. in Alg., **18(6)**(1990), 1971–1988.

[3] Osofsky, B.L. and Smith, P.F., *Cyclic modules whose quotients have all complement submodules direct summands*, J. Alg., **139**(1991), 342–354.

[4] Camillo, V. and Yousif, M.F., *CS−modules with ACC or DCC on essential submodules*, Comm. in Alg., **19(2)**(1991), 655–662.

[5] Li, M.S. and Zelmanowitz, J.M., *On the generalizations of injectivity*, Comm. in Alg., **16(13)**(1988), 483–491.

[6] Kasch, F., *Modules and Rings*, Academic Press, London, 1982.

[7] Muller, B.J. and Tariq Rizvi, S., *On injective and quasi-continuous modules*, J. Pure and Appl. Alg., **28**(1983), 197–210.

[8] Ara, P. and Park, J.K., *On continuous semiprimary rings*, Comm. in Alg., **19(7)**(1991), 1945–1957.

[9] Chatters, A.W. and Hajarnavis, C. R. , *Rings in which every complement right ideal is a direct summand*, Quart. J. Math. Oxford Ser. 2, **28**(1977), 61–80.

Artin 半单环的特征

刘 仲 奎

(西北师范大学数学系，兰州 730070)

摘　要

本文证明了如下结果：环R是Artin半单的当且仅当存在一个基数c, 使得任意左R-模是一个连续模和一个c-限制的ES-模的直和，也当且仅当存在一个基数c, 使得任意左R-模是一个拟投射模和一个c-限制的ES-模的直和.

关键词 Artin 半单环，连续模，c-限制ES-模.

Journal of Mathematical Research & Exposition
Vol.14, No.4, Nov. 1994

A Characterization of Artinian Semisimple Rings *

Liu Zhongkui
(Dept. of Math., Northwest Normal University Lanzhou, 730070)

Abstract It is proved that R is Artinian semisimple if and only if there exists a cardinal c such that every left R-module is the direct sum of a continuous module and a c-limited $ES-$module if and only if there exists a cardinal c such that every left R-module is the direct sum of a quasi-projective module and a c-limited $ES-$module.

Keywords Artinian semisimple ring, continuous module, c-limited ES-module.

In the following, R will always stand for a ring with an identity. It is well-known that R is Artinian semisimple if and only if every left R-module is injective if and only if every left R-module is projective. In this note, we prove that R is Artinian semisimple if and only if there exists a cardinal c such that every left R-module is the direct sum of a continuous module and a c-limited $ES-$module if and only if there exists a cardinal c such that every left R-module is the direct sum of a quasi-projective module and a c-limited $ES-$module.

Let M be a left R-module. Recall that M is called $CS-$module provided every submodule of M is essential in a direct summand of M, or equivalently, every maximal essential extension of a submodule of M is a direct summand of M (see [3] or [4]). M is called continuous if M is a $CS-$module and every submodule isomorphic to a direct summand is a direct summand (see [5] or [9]). A submodule N of a module M is said to be closed in M if N has no proper essential extensions in M.

Lemma 1([5]) *Quasi-injective R-modules are continuous but not conversely.*

Lemma 2([5]) *The following are equivalent for a left R-module M.*

(i) M is continuous.

(ii) If $A \oplus C < M$ and $f : A \oplus C \longrightarrow M$ is a homomorphism with Imf closed in M and $\mathrm{Ker} f = C$, then there exists $g \in \mathrm{End}_R M$ extending f.

A left R-module M is called an ES-module if the socle $\mathrm{Soc}(M)$ is an essential submodule of M (see C.Faith [1]). For example, if M is finitely cogenerated then M is an $ES-$module (see [6]). If R is a PF-ring (that is, R is an injective cogenerator in R-Mod), then $_RR$ is an ES-module. For other examples and results of ES-modules we refer the reader to [1].

*Received Dec. 24, 1992.

Let c be any cardinal. Following [2], a left R-module M will be called c-limited provided every direct sum of non-zero submodules of M contains at most c direct summands (see [2]). For Example, if M has finite uniform dimension then M is $\aleph_0$-limited. For any right R-module M, it is clear that M is c-limited where $c = |M|$. If M is c-limited, then it is easy to see that for every cardinal c' with $c' > c$, M is c'-limited. For other examples of c-limited left R-modules we refer the reader to [2].

In [2], it is proved that R is Artinian semisimple if and only if there exists a cardinal c such that every left R-module is the direct sum of an injective module and a c-limited module if and only if there exists a cardinal c such that every left R-module is the direct sum of a projective module and a c-limited module. Here we have:

Theorem 3 *The following are equivalent for a ring R.*

(i) R is Artinian semisimple.

(ii) There exists a cardinal c such that every left R-module is the direct sum of a continuous module and a c-limited $ES-$module.

(iii) There exists a cardinal c such that every left R-module is the direct sum of a quasi-projective module and a c-limited $ES-$module.

Proof The implications (i)$\Rightarrow$ (ii) and (i)$\Rightarrow$ (iii) are clear.

(ii)$\Rightarrow$ (i). Suppose that there exists a cardinal c such that every left R-module is the direct sum of a continuous module and a c-limited $ES-$module. Obviously, we can assume that c is an infinite cardinal.

Suppose that M is a left R-module. It is enough to show that M is injective. Denote the injective envelope of M by $E(M)$. Set $N = M \oplus E(M)$.

Let $\{S_y | \ y \in J\}$ denote a collection of representatives of the isomorphism classes of simple left R-modules and let $S = \oplus_{y\in J} S_y$. Let K be an index set with $|K| \geq c$, and, for each z in K, let $T_z = S$, define

$$T = \oplus_{z\in K} T_z.$$

Let I be an index set with $|I| > |E(T)|$, where $E(T)$ is the injective envelope of left R-module T. For each x in I let $N_x = N$, and

$$F = \oplus_{x\in I} N_x.$$

By assumption, there exists a continuous left R-module A and a c-limited ES-module B such that $F = A \oplus B$. Note that Soc(B) is a direct sum of at most c simple submodules of B, it is clear that there exists a monomorphism f : Soc$(B) \longrightarrow T$. Thus we obtain a homomorphism $g : B \longrightarrow E(T)$ such that $g|_{Soc(B)} = f$. Since B is an $ES-$module, Soc(B) is an essential submodule of B, which implies that g is a monomorphism. Thus we have $|B| \leq |E(T)|$.

For each $b \in B$, there exists a finite subset $I(b)$ of I such that $b \in \oplus_{x\in I(b)} N_x$. Let $I' = \cup_{b\in B} I(b)$. If $|B| < \infty$, then $|I'| < \infty$. Thus $|I'| \leq |E(T)|$. Now suppose that $|B|$ is an infinite cardinal. Then $|I'| \leq |B| \leq |E(T)|$. Set $I" = I - I''$. From the construction of I it follows that $|I| > |E(T)|$, and thus $I'' \neq \emptyset$. Now let

$$G = \oplus_{x\in I'} N_x, \quad H = \oplus_{x\in I''} N_x.$$

Then we have $F = G \oplus H = A \oplus B$, and $B \leq G$. Thus it follows by modularity that $G = (A \cap G) \oplus B$. So $F = A \oplus B = (A \cap G) \oplus B \oplus H$, which implies that $A \simeq (A \cap G) \oplus H$. Since A is continuous, it follows that H is continuous, too, by Theorem 13 of [7]. Thus $N = M \oplus E(M)$, a direct summand of H, is continuous.

Consider the following diagram

$$\begin{array}{ccc} M & \xrightarrow{h} & E(M) \\ \epsilon_1 \downarrow & & \downarrow \epsilon_2 \\ M \oplus E(M) & \xleftarrow{h'} & M \oplus E(M) \end{array}$$

where ϵ_1 and ϵ_2 are the natural injections and h is the inclusion map. It is clear that $\mathrm{Im}\epsilon_1 = (M, 0)$. Suppose that there exists a submodule L of $M \oplus E(M)$ which is an essential extension of $(M, 0)$. Set $L_1 = \{(0, n) |$ there exists $m \in M$ such that $(m, n) \in L\}$. Since $(0, n) = (m, n) - (m, 0) \in L$, L_1 is a submodule of L. If $L_1 \neq 0$, then $L_1 \cap (M, 0) \neq 0$. This is a contradiction. Thus $L_1 = 0$, which implies that $L = (M, 0)$. This means that $(M, 0)$ is closed in $M \oplus E(M)$. Thus, by Lemma 2, there exists an endomorphism h' of $M \oplus E(M)$ such that $h'\epsilon_2 h = \epsilon_1$. Now if $\pi_1 : M \oplus E(M) \longrightarrow M$ is the natural projection, then we have $\pi_1 h' \epsilon_2 h = \pi_1 \epsilon_1 = 1_M$. This means that $h : M \longrightarrow E(M)$ is split. Thus M is injective, and we are done.

(iii)⇒ (i). Suppose that there exists a cardinal c such that every left R-module is the direct sum of a quasi-projective module and a c-limited $ES-$module. Let M be a cyclic left R-module. Then we have the following exact sequence:

$$0 \longrightarrow \mathrm{Ker} f \longrightarrow R \stackrel{f}{\longrightarrow} M \longrightarrow 0.$$

Set $N = M \oplus R$. Let I be an index set. For each x in I let $N_x = N$, and set

$$F = \oplus_{x \in I} N_x.$$

By analogy with the proof of $(ii) \Rightarrow (i)$, set $S = \oplus S_y$ and $T = \oplus T_z$. Also we can made $|I| > |E(T)|$. By assumption, there exists a quasi-projective left R-module A and a c-limited ES-module B such that $F = A \oplus B$. Since every direct summand of a quasi-projective left R-module is quasi-projective, by analogy with the proof of (ii)⇒ (i), we obtain that $N = M \oplus R$ is quasi-projective.

Now consider the following diagram

$$\begin{array}{ccc} M \oplus R & \xrightarrow{\pi_1} & M \\ g \downarrow & & \uparrow f \\ M \oplus R & \xrightarrow{\pi_2} & R \end{array}$$

where π_1 and π_2 are the natural projections. By the quasi-projectivity of $M \oplus R$, there exists a homomorphism $g \in \mathrm{End}(M \oplus R)$ such that $\pi_1 = f\pi_2 g$. Thus we have $f\pi_2 g\epsilon_1 =$

$\pi_1\epsilon_1 = 1_M$, where ϵ_1 is the natural injection from M to $M \oplus R$. This means that f is split. Thus M is projective.

Therefore every cyclic left R-module is projective. So R is Artinian semisimple, and we are done.

Acknowledgement *This research was supported by NSF of Gansu, China.*

References

[1] Faith, C., *Annihilator ideals, associated primes and Kasch-McCoy Commutative rings*, Comm. in Alg., **19(7)**(1991), 1867–1892.

[2] Huynh, D.Van and Smith, P.F., *Some rings characterised by their modules*, Comm. in Alg., **18(6)**(1990), 1971–1988.

[3] Osofsky, B.L. and Smith, P.F., *Cyclic modules whose quotients have all complement submodules direct summands*, J. Alg., **139**(1991), 342–354.

[4] Camillo, V. and Yousif, M.F., *CS−modules with ACC or DCC on essential submodules*, Comm. in Alg., **19(2)**(1991), 655–662.

[5] Li, M.S. and Zelmanowitz, J.M., *On the generalizations of injectivity*, Comm. in Alg., **16(13)**(1988), 483–491.

[6] Kasch, F., *Modules and Rings*, Academic Press, London, 1982.

[7] Muller, B.J. and Tariq Rizvi, S., *On injective and quasi-continuous modules*, J. Pure and Appl. Alg., **28**(1983), 197–210.

[8] Ara, P. and Park, J.K., *On continuous semiprimary rings*, Comm. in Alg., **19(7)**(1991), 1945–1957.

[9] Chatters, A.W. and Hajarnavis, C. R., *Rings in which every complement right ideal is a direct summand*, Quart. J. Math. Oxford Ser. 2, **28**(1977), 61–80.

Artin 半单环的特征

刘仲奎

(西北师范大学数学系，兰州730070)

摘 要

本文证明了如下结果：环R是Artin半单的当且仅当存在一个基数c，使得任意左R-模是一个连续模和一个c-限制的ES-模的直和，也当且仅当存在一个基数c，使得任意左R-模是一个拟投射模和一个c-限制的ES-模的直和.

关键词 Artin 半单环，连续模，c-限制ES-模.