半值环的两个交换结果

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摘　要

定理1　设R 是半值环，n 为固定的正整数，如果R 满足条件：$\forall x,y \in R$, 存在依赖于x,y 的两个字$k(X,Y),t(X,Y)$, 其中$|k|_X > 1$, $|t|_X = 1$, $|k|_Y \geq |t|_Y$, $|t|_Y \leq n$, 使$k(x,y)-t(x,y) \in I(R)$, 则R 是交换环.

定理2　设R 是半值环，如果R 满足条件：$\forall x,y \in R$, 存在正整数$m=m(x,y)>1$, $n=n(y)$, 使得$(x^ny)^m - x^ny \in I(R)$, 则R 是交换环.

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Two Commutativity Results for Semiprime Rings *

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Abstrict In this paper, we study commutativity conditions for semiprime rings by means of words in non-commutative indeterminates X and Y, and obtain two commutativity results.

Keywords semiprime rings, Jacobson radicals, quasiregular.

Throughout this paper, R will represent an associative ring (may be without unity) with center $Z(R), J(R)$ the Jacobson radical of R. We say $k(X,Y)$ a word in non-commutative indeterminates X and Y if $k(X,Y)$ has the following form

$$k(X,Y)=X^{i_1}Y^{j_1}X^{i_2}Y^{j_2}\cdots X^{i_m}Y^{j_m},$$

where $i_s\geq 0$, $j_s\geq 0$, $s=1,2,\ldots,m$, $\sum_{s=1}^{m}(i_s+j_s)>0$. We also write $|k|_X$ for $\sum_{s=1}^{m}i_s$, $|k|_Y$ for $\sum_{s=1}^{m}j_s$ and $|k|$ for $|k|_X+|k|_Y$.

In 1986, Quadri, Ashraf and Khan [5] proved that a semiprime ring R satisfying $(xy)^2-xy\in Z(R)$ for all $x,y\in R$ is commutative. In this direction Guo Xiuzhan [4] proved the following.

Theorem *Let R be a semiprime ring satisfying $(x^my)^n-x^my\in Z(R)$ for all $x,y\in R$, where m,n are fixed positive integers, $n>1$, then R is commutative.*

In this paper, we prove the following.

Theorem 1 *Let R be a semiprime ring satisfying that for each $x,y\in R$, there exist two words $k(X,Y)$ and $t(X,Y)$ depending on x,y, with $|k|_X>1$, $|t|_X=1$, $|k|_Y\geq|t|_Y$, $|t|_Y\leq n$, where n is a fixed positive integer, such that*

$$k(x,y)-t(x,y)\in Z(R),$$

then R is commutative.

Theorem 2 *Let R be a semiprime ring satisfying that for each $x,y\in R$, there exist two*

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positive integers $m = m(x, y) > 1$, $n = n(y)$ such that $(x^n y)^m - x^n y \in Z(R)$, then R is commutative.

For the proofs of Theorem 1, 2, we need some lemmas.

Lemma 1[1] *Let R be a semiprime ring, $0 \neq a \in Z(R)$ and $x \in R$. If $ax \in Z(R)$, then $x \in Z(R)$.*

Lemma 2[2] *Let R be a ring satisfying that for any $x \in R$, there exists a polynomial $p(t)$ with integral coefficients such that $x^2 p(x) - x \in Z(R)$, then R is commutative.*

Lemma 3[3] *If R has a non-zero nil one-sided ideal with a limited index, then R has a non-zero nilpotent ideal.*

Lemma 4[4] *If R has an ideal I which has no non-zero nilpotent elements, such that R/I and I are both commutative, then R is commutative.*

Proof Let $M = \{x \in R | IxI = (0)\}$. Since I has no non-zero nilpotent elements and $MI \subseteq I$, $(M \cap I)^2 \subseteq MI$, $(MI)^2 = (0)$, we have $M \cap I = (0)$.

Let $x, y \in R$, R/I is commutative by assumption, hence $xy - yx \in I$. On the other hand, for any $a, b \in I$, we have $axyb = ybax = aybx = bayx = ayxb$. Hence $xy - yx \in M$.

It follows that $xy = yx$ for all $x, y \in R$.

Lemma 5 *Let R be a ring satisfying the condition of Theorem 1. If $J(R) = (0)$ then R is commutative.*

Proof Because R is isomorphic to a sub-direct sum of primitive rings, and the condition of Theorem 1 is inherited by homomorphic images. It is clear that it suffices to prove Lemma 5 with assumption that R is a primitive ring.

Suppose R is not a division ring, it follows from the density theorem for primitive rings that there exists a homomorphism of a subring of R onto a complete matrix ring Δ_2 over a division ring Δ. Let $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $k(x, y) - t(x, y) \notin Z(R)$ for all words $k(X, Y)$ and $t(X, Y)$ with $|k|_X > 1$, $|t|_X = 1$. Hence R is a division ring. For each $x \in R$, put $y = 1$. It follows from the condition of Theorem 1 that there exists a integer $m = m(x) > 1$ such that $x^m - x \in Z(R)$. By Lemma 2, R is commutative.

Lemma 6 *Let R be a prime ring satisfying the hypothesis of Theorem 1, then R has no non-zero nilpotent elements.*

Proof If $J(R) = (0)$, then R is commutative by Lemma 5. Hence R has no non-zero nilpotent elements. Suppose $J(R) \neq (0)$, if $Z(R) = (0)$, then for each $x \in J(R)$ we have

$$|x|^{|t|}(1 - x^{|k|-|t|}) = x^{|t|} - x^{|k|} = 0,$$

but $x^{|k|-|t|}$ is qusi-regular. Hence

$$x^{|t|} = 0, \quad |t| \leq n + 1.$$

This means $J(R)$ is non-zero nil ideal with limited index. It follows from Lemma 3 that R has a non-zero nilpotent ideal, but R is a prime ring, this is impossible. Hence $Z(R) \neq (0)$.

Let $x \in R$, $x^2 = 0$. Take $0 \neq a \in Z(R)$, then there exist two words $k(X,Y)$ and $t(X,Y)$ with $|k|_X > 1$, $|t|_X = 1$ such that

$$-a^{|t|_Y} x = k(x,a) - t(x,a) \in Z(R).$$

By Lemma 1 and $a^{|t|_Y} \neq 0$, we have $x \in Z(R)$. Hence $x = 0$.

Lemma 7 *Let R be a prime ring satisfying the hypothesis of Theorem 2, then R has no non-zero nilpotent elements.*

Proof Let $a \in R$ such that $a^2 \neq 0$. By hypothesis we have $-(ax)^n a = [(ax)^n a]^m - (ax)^n a \in Z(R)$ for all $x \in R$, where $n = n(a)$ is only dependent on the element a. Thus, $(ax)^{n+2} = 0$.

If $aR \neq 0$, then R has a non-zero nilpotent ideal by Lemma 3, which contradicts to the fact that R is a prime. Thus, $aR = 0$, hence $aRa = (0)$, this implies that $a = 0$.

Proof of Theorem 1 It suffices to show that a prime ring R satisfying the hypothesis of Theorem 1 is commutative.

If $J(R) = (0)$, then Theorem 1 is correct by Lemma 5. Suppose $J(R) \neq (0)$. Let $x \in J(R)$, $x \neq 0$, then there exist two words $k(X,Y)$ and $t(X,Y)$ depending on x, with $|k| > |t|$ such that $\lambda = x^{|k|} - x^{|t|} \in Z(R)$. It is clear $\lambda \neq 0$ by Lemma 6. For x, λ, there exist two words $k_1(X,Y)$ and $t_1(X,Y)$ with $|k_1|_X > 1$, $|t_1|_X = 1$, $|k_1|_Y \geq |t_1|_Y$ such that $\lambda^{|k_1|_Y} x^{|k_1|_X} - \lambda^{|t|_Y} x \in Z(R)$. It follows from Lemma 1 that

$$\lambda^{|k_1|_Y - |t_1|_Y} x^{|k_1|_X} - x \in Z(R).$$

Thus, $J(R)$ is commutative by Lemma 2. Since $R/J(R)$ is commutative, we have R is commutative by Lemma 4.

Using Lemma 1, 2, 3, 4, 7, we can prove Theorem 2 in a similar way.

Corollary 1 Let R be a semiprime ring and m a fixed integer, $m \geq 2$. If for each $x_1, x_2, \cdots, x_m \in R$, there exists a integer $t = t(x_1, x_2, \cdots, x_m) > 1$ such that $(x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(m)})^t - x_1 x_2 \cdots x_m \in Z(R)$, where $\sigma(1), \sigma(2), \cdots, \sigma(m)$ is a permutation of $1, 2, \cdots, m$, then R is commutative.

Corollary 2 Let R be a semiprime ring and s a fixed integer. If for each $x, y \in R$, there exists two integers $m = m(x,y) > 1$, $n = n(x,y) \geq s$ such that $(x^n y)^m - x^s y \in Z(R)$, then R is commutative.

References

[1] V. Gupta, Acta Math. Acad. Sci. Hungar., **36**(1980), 233–236.
[2] I.N. Herstein, Amer. J. Math., **75**(1953), 864–871.
[3] Xie Bangjie, J. of Natural Science of Northeastern People's University, **1**(1955), 13–14.
[4] Guo Xiuzhan, J. of Mathematical Research and Exposition, **4**(1991), 575–578.
[5] M.A. Quadri, M. Ashraf and M.A. Khan, Bull. Austral. Math. Soc., **33**(1986), 71–73.