which implies

$$(\beta+1)^{m-1}=m. \tag{3}$$

On the other hand, for $m \geq 2$, $\beta+1 \geq 2$, we can prove easily by induction on m that the inequality

$$(\beta+1)^{m-1} \geq m \tag{4}$$

and equality holds only for $m=2$ and $\beta+1=2$. Then owing to (3), it follows that $m=2$ and $\beta=1$. Furthermore, by (2) we obtain $\alpha=3$. The proof is completed.

Remark Our theorem, to some extent, is related to the following theorm of J. Schaffer, but these two results are independent of each other. In [2] Schaffer proved that for fixed integers $p>0$ and $q>1$ the equation.

$$\sum_{j=1}^{n} j^p = m^q \tag{5}$$

has an infinite number of solutions in positive integers m and n only in the cases (i) $p=1$, $q=2$; (ii) $p=3, q=2$ or 4; (iii) $p=5, q=2$. Recently, Schaffer's work has been generalized in [3] and [4].

References

[1] W.J. LeVeque, *On the equation $a^x-b^y=1$*, Amer. J. Math., **74**(1952), 325—331.

[2] J.J. Schaffer, *The equation $1^p+2^p+\cdots+n^p=m^p$*, Acta. Math., **95**(1956), 155—331.

[3] M. Voorhoeve, K. Gyory and R. Tijdeman, *On the diophantine equation $1^k+2^k+\cdots+x^k+R(x)=y^z$*, Acta. Math., **143**(1979), 1—8.

[4] K. Gyory, R. Tijdeman and M. Voorhoeve, *On the equation $1^k+2^k+\cdots+x^k=y^k$*, Acta. Arith., **37**(1980), 233—240.

W.J. LeVeque 的一个定理的注记

余红兵

(中国科技大学数学系，合肥 230026)

摘　要

本文证明了：对固定的正整数α,β,m, 其中$m\geq 2$, 若方程$\sum_{j=1}^{n} j^\alpha=(\sum_{j=1}^{n} j^\beta)^m$有无穷多个正整数解$n$, 则$m=2,\alpha=3$及$\beta=1$. 这推广了LeVeque 的一个结果.

关键词:　不定方程，恒等式.

Journal of Mathematical Research & Exposition
Vol.14, No.4, Nov. 1994

A Note on a Theorem of W.J. LeVeque *

Yu Hongbing
(Dept. of Math., University of Science and Technology of China, Hefei Anhui, China)

Abstract It is shown that for fixed positive integers α, β, m and $m \geq 2$ the equation $\sum_{j=1}^{n} j^{\alpha} = (\sum_{j=1}^{n} j^{\beta})^{m}$ has infinitely many solutions in positive integers n only when $m = 2, \alpha = 3$ and $\beta = 1$, which generalizes a result of LeVeque.

Keywords Diophantine equation, Identity.

In [1] LeVeque proved the following intersting result:
If for all n there holds an identity of the form

$$\sum_{j=1}^{n} j^{\alpha} = (\sum_{j=1}^{n} j^{\beta})^{m} \tag{1}$$

with fixed positive integers α, β, m and $m \geq 2$, then $m = 2, \alpha = 3$ and $\beta = 1$.

LeVque's proof of this result rests on his works on Diophantine equations $a^{x} + 1 = (a^{y} + 1)^{z}$ which is more difficult. In this note, we prove the following theorem which generalizes LeVeque's result mentioned above.

Theorem *For fixed positive integers α, β, m and $m \geq 2$, if the equation (1) has infinitely many solutions in positive integers n, then $m = 2, \alpha = 3$ and $\beta = 1$.*

Proof The proof is very simple and it is "analytic", not "arithmetical ".

In fact, it is well known that for fixed $k > 0$ we have

$$\sum_{j=1}^{n} j^{k} \sim \frac{1}{k+1} n^{k+1}$$

as $n \to \infty$. Since there exists arbitrary large n such that equation (1) holds, the left side of (1) is $\sim \frac{1}{\alpha+1} n^{\alpha+1}$ and the right side is $\sim (\frac{1}{\beta+1} n^{\beta+1})^{m}$, therefore we get

$$\frac{1}{\alpha+1} = (\frac{1}{\beta+1})^{m} \quad \text{and} \quad \alpha + 1 = (\beta+1)m, \tag{2}$$

*Received May 16, 1992. The Project Supported by Youth Science Foundation of the USTC.

which implies

$$(\beta+1)^{m-1}=m. \tag{3}$$

On the other hand, for $m \geq 2$, $\beta+1 \geq 2$, we can prove easily by induction on m that the inequality

$$(\beta+1)^{m-1} \geq m \tag{4}$$

and equality holds only for $m=2$ and $\beta+1=2$. Then owing to (3), it follows that $m=2$ and $\beta=1$. Furthermore, by (2) we obtain $\alpha=3$. The proof is completed.

Remark Our theorem, to some extent, is related to the following theorm of J. Schaffer, but these two results are independent of each other. In [2] Schaffer proved that for fixed integers $p>0$ and $q>1$ the equation.

$$\sum_{j=1}^{n} j^{p}=m^{q} \tag{5}$$

has an infinite number of solutions in positive integers m and n only in the cases (i) $p=1$, $q=2$; (ii) $p=3, q=2$ or 4; (iii) $p=5, q=2$. Recently, Schaffer's work has been generalized in [3] and [4].

References

[1] W.J. LeVeque, *On the equation* $a^x-b^y=1$, Amer. J. Math., **74**(1952), 325—331.

[2] J.J. Schaffer, *The equation* $1^p+2^p+\cdots+n^p=m^p$, Acta. Math., **95**(1956), 155—331.

[3] M. Voorhoeve, K. Gyory and R. Tijdeman, *On the diophantine equation* $1^k+2^k+\cdots+x^k+R(x)=y^z$, Acta. Math., **143**(1979), 1—8.

[4] K. Gyory, R. Tijdeman and M. Voorhoeve, *On the equation* $1^k+2^k+\cdots+x^k=y^k$, Acta. Arith., **37**(1980), 233—240.

W.J. LeVeque 的一个定理的注记

余红兵
(中国科技大学数学系, 合肥 230026)

摘 要

本文证明了: 对固定的正整数α,β,m, 其中$m\geq 2$, 若方程$\sum_{j=1}^{n} j^{\alpha}=(\sum_{j=1}^{n} j^{\beta})^{m}$有无穷多个正整数解$n$, 则$m=2,\alpha=3$及$\beta=1$. 这推广了LeVeque 的一个结果.

关键词: 不定方程, 恒等式.