I To get a regular GC^1 surface, the lengthes of vectors $b_{i,j} - b_i$ and $b_{j,i} - b_j$ should not be too long. It is better to chose $b_{i,j}$ and $b_{j,i}$ such that

$$\|b_{i,j} - b_i\|,\ \ \|b_{j,i} - b_j\| < \frac{1}{2}\|v_j - v_i\|.$$

II If we take $\{b_{i,j}, b_{j,i}\}_{e_{i,j}\in\delta}$ as an element in $\mathbb{R}^{4N}$ (the reason that we can do so is that $\{b_{i,j} - b_i\}_j \in \pi_i$ and $\{b_{j,i} - b_j\}_j \in \pi_j$ are coplanar vectors, respectively), then

$$m(\bigcup_{i,j} E_{i,j}) = 0,$$

where N is the number of edges in Δ, mA is the measure of the collection A, and $E_{i,j} := \{X \in \mathbb{R}^{4N} :\ J_{i,j}(X) := J_{i,j} = 0,\ \ J_{i,j}$ is defined in (13)$\}$. So the constraints of the choosing of the control vectors $\{b_{i,j}, b_{j,i}\}_{e_{i,j}\in\delta}$ are very mild.

References

[1] R.E. Barnhill, *Surfaces in computer aided geometric design: a survey with new results*, CAGD **2**(1985), 1—17.
[2] W. Boehm, *Cubic B-spline surves and surfaces in computer aided geometric design*, Computing **19**(1977), 29—34.
[3] W. Boehm, *cubics: a survey*, Computer Graphics and Image Processing, **19**(1982), 201–226.
[4] R. Clough and J.Tocher, *Finite elslent stiffness matrices for analysis of plates in blending*, In Proceedings of Conference on Matrix Methods in Structural Analysis, 1965.
[5] W. Dahmen, *Smooth Piecewise Quadric Surfaces*, Mathematical Methods in Computer Aided Geometric Design, Tom Lyche and L. Schümaker (eds.), 181—193.
[6] G. Farin, *Bezier Polynomials over triangles and the construction of piecewise C^r Polynomials*, Technical Report TR/91 Dept. Mathematics, Brunel Univ. Uxbridge, England, 1980.
[7] G. Farin, *Smooth interpolation top scattered 3-D date*, In R. Barnhill and W. Boehm, editors, Surfaces in Computer Aided Geometric Design, North-Holland, 1982.
[8] G. Farin (ed.), *Geometric Modeling: Algorithms and New Trends*, SIAM, 1987.
[9] D.Y. Liu, *A geometric condition for smoothness between adjacent Bezier surface patches*, ACTA Math. Appl. Sinica, Vol.9, No.4, Oct., 1986.
[10] B.Piper, *Visually smooth interpolation with triangular Bezier patches*, In G.Farin, editor, Geometric Modeling: Algorithms and New Trends, SIAM, 1987, 221—234.
[11] X.Q. Shi and T.J. Wang, *The GC^1 quintic interpolant on arbitrary triangular mesh*, J. of Mathematical Research & Exposition, **4**(1992), 583–588.

空间网格上的三次GC'插值格式

施 锡 泉　　　　王 天 军
(大连理工大学数学科学研究所, 116024) (北京航空航天大学)

摘　要

本文得到了由三次参数多项式构造的GC'插值格式，该插值格式定义在由空间三角形和空间四边形构成的空间网格上，并通过该网格的所有网点，同时在每个网点处以事先给定的平面为切平面.

Journal of Mathematical Research & Exposition
Vol.14, No.4, Nov. 1994

A GC^1 Cubic Interpolant on a Space Mesh *

Shi Xiquan
(Inst. of Math. Science, Dalian Univ. of Tech.)

Wang Tianjun
(Beijing Univ. of Aeronautics & Astronautics

Abstract In this note we derive an interpolation scheme for constructing GC^1 surfaces by cubic parameterized polynomials. This scheme can be used to construct GC^1 surfaces over the so called space meshes which are composed of space triangles and space quadrangles. As a special case, it can be used to construct GC^1 surfaces over triangular meshes. In addition, the results in this note was published as LETTERS in Chinese Science Bulletin (Vol. 36, No. 20, Oct., 1991).

Keywords free surface, interpolant, space mesh.

1. Introduction

It is well-known that Surfaces in Computer Aided Geometric Design (SCAGD) is an important branch of CAGD. It has wide-ranging applications, including the design of cars, ships, aeroplanes, and many others objects, and the modeling of human organs and robots. So this field is receiving more and more interests of scientists and many interesting results have been obtained. For the results of this aspect the reader can read R.E. Barnhill's [1] survey. B.R. Piper obtained an interpolation scheme for constructing sufaces over a triangular mesh by using quartic parameterized polynomials in the case of Clough-Tocher split and he illustrated by an example that cubic polynomials are not enough to construct a GC^1 interpolating surface to the given position data. Under the same conditions, however, our result shows that cubic parameterized polynomials are sufficient to construct such a surface under very mild constraints. At the same time, we have also considered the construction of GC^1 surface on space quadrangles and furthermore, on a so called space mesh which is composed of space triangles and space quadrangles. All polynomials in this paper will be parameterized and vector valued, and they are represented in Bernstein-Bezier form and we assume that the reader is familiar with the associated theory, see G. Farin [6].

*Received Sep. 9, 1992. The Project is Partly Supported by the Natrual Science Foundation of China and Partly Supported by Alexander Von Homboldt-Stiftung, and was Compteled During the First Author Visiting Inst. of Math., Freie Univestät Berlin.

We would like to point out here that the surfaces constructed in this paper interpolate the given position vectors, and in correspondence to each given position vector these surfaces take the given planes as their tangent planes. The scheme introduced in this paper can be used to construct ball-like surfaces, closed surfaces and other regular surfaces.

We present a scheme of constructing GC^1 surfaces on a triangle by using cubic patches in Section 2. The problem for constructing GC^1 surfaces on space quadrangles will be discussed in Section 3. Finally, the method of constructing a global GC^1 surface will be described in Section 4.

2. GC^1 surfaces on a triangle

In this section, we will study the GC^1 joint of two polynomial patches on adjacent triangles. Then we will consider the GC^1 joint of cubic patches on a triangle in the case of Clough-Tocher split.

Definition 1 *A collection Δ of vertices, edges, triangles and space quadrangles is called a space mesh if it satisfies the following conditions:*

1). If an edge, or a triangle, or a space quadrangle belongs to Δ, then all its faces are again in Δ, respectively, and any vector or edge in Δ belongs to a triangle or a space quadrangle in Δ.

2). The joint of any two elements in Δ is again in Δ and it is a common face of each.

3). For each edge $e \in \Delta$, it holds $N(e) \le 2$, where $N(e)$ is the sum of the numbers of triangles and space quadrangles in Δ which take e as their a common edge.

4). Planar quadrangles in Δ are convex.

For an edge $e \in \Delta$, if there is only one triangle or space quadrangle with e as its edge, then it is called a boundary edge; otherwise it is called an inner edge. A vertex of a boundary edge is called a boundary vertex; otherwise if all the edges which take this vertex as a common end-point are inner edges, then it is called an inner vertex. If a triangle and space quadrangle or two triangles or two space quadrangles have a common edge, then they are called adjacent elements. Two polynomial patches (or surfaces) are called adjacent if they are on two adjacent elements, respectively.

In this paper, we will use the following formula

$$P(v) = \sum_{i\in\Gamma_\delta} (b_i u_i^3 + 3 \sum_{i\neq j\in\Gamma_\delta} b_{i,j} u_i^2 u_j) + 6 b_\delta u_0 u_1 u_2 \tag{1}$$

to denote a cubic polynomial patch on a triangle $\delta = [v_0, v_1, v_2]$, where and throughout the paper, $\Gamma_\delta = \{i | v_i \text{ is a vertex of } \delta\}$, $b_\delta, b_i, b_{i,j} \in IR^3$ are called control vectors, respectively, and (u_0, u_1, u_2) is the barycentric coordinates of v with respect to δ.

Let $\delta_1 = [v_0, v_1, v_2]$ and $\delta_2 = [v_0, v_1, v_3]$ be two adjacent triangles. By

$$P_m(v) = \sum_{i\in\Gamma_{\delta_m}} (b_i u_i^3 + 3 \sum_{i\neq j\in\Gamma_{\delta_m}} b_{i,j} u_i^2 u_j) + 6 b_{\delta_m} u_0 u_1 u_{1+m} \tag{2}$$

we denote the cubic polynomial patch on δ_m $(m = 1, 2)$, respectively.

It is well known that the necessary and sufficient condition of GC^1 joint between $P_1(v)$ and $P_2(v)$ is that the following condition holds for $v \in \delta_1 \cap \delta_2$

$$\det(D_{0,1}P_1(v), D_{0,2}P_1(v), D_{0,3}P_2(v)) = 0, \tag{3}$$

where $D_{0,i}f = (v_i - v_0)\cdot \mathrm{grad} f$ is the directional derivative of f along vector $v_i - v_0$, $X \cdot Y$ is the inner product of X and Y, and det A is the determinant of matrix A.

Especially, if for $v \in \delta_1 \cap \delta_2$ and some real constants $\{\alpha_i, \beta_i\}_{i=1}^3$,

$$(\alpha_1 u_0 + \beta_1 u_1)D_{0,1}P_1(v) + (\alpha_2 u_0 + \beta_2 u_1)D_{0,2}P_1(v) + (\alpha_3 u_0 + \beta_3 u_1)D_{0,3}P_2(v), \tag{4}$$

then $P_1(v)$ and $P_2(v)$ is GC^1 joint, where (u_0, u_1) is the barycentric coordinates of v with respect to $v_0v_1 = \delta_1 \cap \delta_2$, and α_i and β_i are notall zeros. It is easy to see that equation (4) is only a sufficient condition of GC^1 joint of $P_1(v)$ and $P_2(v)$.

According to equation (2), we have

$$\begin{aligned}
D_{0,1}P_1(v) &= 3((b_{0,1} - b_0)u_0^2 + 2(b_{1,0} - b_{0,1})u_0u_1 + (b_1 - b_{1,0})u_1^2), \\
D_{0,2}P_1(v) &= 3((b_{0,2} - b_0)u_0^2 + 2(b_{\delta_1} - b_{0,1})u_0u_1 + (b_{1,2} - b_{1,0})u_1^2), \\
D_{0,3}P_1(v) &= 3((b_{0,3} - b_0)u_0^2 + 2(b_{\delta_2} - b_{0,1})u_0u_1 + (b_{1,3} - b_{1,0})u_1^2).
\end{aligned}$$

Thus, equation (4) is equivalent to

$$\begin{aligned}
&\alpha_1(b_{0,1} - b_0) + \alpha_2(b_{0,2} - b_0) + \alpha_3(b_{0,3} - b_0) = 0, \\
&\alpha_2(b_{\delta_1} - b_{0,1}) + \alpha_3(b_{\delta_2} - b_{0,1}) + A = 0, \\
&\beta_2(b_{\delta_1} - b_{0,1}) + \beta_3(b_{\delta_2} - b_{0,1}) + B = 0, \\
&\beta_1(b_1 - b_{1,0}) + \beta_2(b_{1,2} - b_{1,0}) + \beta_3(b_{1,3} - b_{1,0}) = 0,
\end{aligned} \tag{5}$$

where $A = \alpha_1(b_{1,0} - b_{0,1}) + \frac{1}{2}(\beta_1(b_{0,1} - b_0) + \beta_2(b_{0,2} - b_0) + \beta_3(b_{0,3} - b_0))$ and $B = \beta_1(b_{1,0} - b_{0,1}) + \frac{1}{2}(\alpha_1(b_1 - b_{1,0})\alpha_2(b_{1,1} - b_{1,0}) + \alpha_3(b_{1,3} - b_{1,0}))$. According to equation (5), we can set

$$\begin{aligned}
\alpha_2 &= \|(b_{0,1} - b_0) \times (b_{0,3} - b_0)\|, \quad \alpha_3 = \|(b_{0,1} - b_0) \times (b_{0,2} - b_0)\|, \\
\beta_2 &= \|(b_{1,0} - b_1) \times (b_{1,3} - b_1)\|, \quad \beta_3 = \|(b_{1,0} - b_1) \times (b_{1,2} - b_1)\|,
\end{aligned}$$

where $\|X\|$ is the l_2 norm of X, and $X \times Y$ is the vector produce of X and Y. If we assume

$$J := \alpha_2\beta_3 - \alpha_3\beta_2 \neq 0, \tag{6}$$

then according to (5), we have

$$\begin{aligned}
b\delta_1 &= b_{0,1} - \frac{1}{J}(\beta_3 A - \alpha_3 B), \\
b\delta_2 &= b_{0,1} - \frac{1}{J}(\beta_2 A - \alpha_2 B).
\end{aligned} \tag{7}$$

In the following, we will discuss the GC^1 joint of cubic patches on a triangle which is divided into three subtriangles by Clough-Tocher split.

Let $\delta = [v_1, v_2, v_3]$ be a triangle, and v_0 be its barycenter, and

$$P_m(v) = \sum_{i\in\Gamma_{\delta_m}} (b_i u_{m,i}^3 + 3 \sum_{i\neq j\in\Gamma_{\delta_m}} b_{i,j} u_{m,i}^2 u_{m,j}) + 6 b_{\delta_m} u_{m,0} u_{m,m+1} u_{m,m+2}$$

be the cubic patch on the triangle $\delta_m = [v_0, v_{m+1}, v_{m+2}]$, respectively, where $v_4 = v_1, v_5 = v_2$ and $(u_{m,0}, u_{m,m+1}, u_{m,m+2})$ is the barycentric coordinates of $v = u_{m,0}v_0 + u_{m,m+1}v_{m+1} + u_{m,m+2}v_{m+2}$ with respect to δ_m, respectively.

According to equation (3), if there are some positive constants $\{\alpha_i', \beta_i'\}_{i=1}^3$ such that for $v \in \delta_{i+1} \cap \delta_{i+2}$,

$$D_{i,0}P_{i+2}(v) - \alpha_i' D_{i,i+1}P_{i+2}(v) - \beta_i' D_{i,i+2}P_{i+1}(v) = 0, \quad 1 \le i \le 3, \tag{8}$$

then P, $P|_{\delta_i} = P_i$ $(i = 1, 2, 3)$, is a GC^1 smooth surface on δ, where $\delta_{i+3} = \delta_i$, $P_{3+i} = P_i$. Sililary, equation (8) is equivalent to

$$\begin{aligned} b_{i,0} - b_i &= \alpha_i'(b_{i,i+1} - b_i) + \beta_i'(b_{i,i+2} - b_i) \\ b_{0,i} - b_{i,0} &= \alpha_i'(b_{\delta_{i+2}} - b_{i,0}) + \beta_i'(b_{\delta_{i+1}} - b_{i,0}) \quad 1 \le i \le 3 \ , \\ b_0 - b_{0,i} &= \alpha_i'(b_{0,i+1} - b_{0,i}) + \beta_i'(b_{0,i+2} - b_{0,i}) \end{aligned} \tag{9}$$

where $b_{i,3+j} = b_{i,j}$ and $b_{\delta_{i+3}} = b_{\delta_i}$.

If we set $\alpha_i' = \beta_i' = \dfrac{1}{3}$, then equation (9) is equivalent to

$$\begin{aligned} b_{i,0} &= \tfrac{1}{3}(b_i + b_{i,i+1} + b_{i,i+2}), \\ b_{0,i} &= \tfrac{1}{3}(b_{i,0} + b_{\delta_{i+1}} + b_{\delta_{i+2}}), \qquad 1 \le i \le 3 \\ b_0 &= \tfrac{1}{3}(b_{0,1} + b_{0,2} + b_{0,3}). \end{aligned} \tag{10}$$

So we can construct a GC^1 surface on a triangle δ by using three cubic patches as the following:

I. The control vectors $\{b_i, b_{i,i+1}, b_{i,i+2}, b_{\delta_i}\}_{i=1}^3$ are chosen arbitrarily.

II. The other control vectors are determined by equation (10).

In this section we will discuss how to construct a GC^1 surface on a space quadrangle $\delta = [v_1, v_2, v_3, v_4]$ with cubic patches, where, in fact, a surface on δ we mean a surface defined on the following surface

$$Q: \sum := \{(x, y, z) = f(a, b), \ \ 0 \le a, b \le 1\},$$

where $f(a, b) = (1-a)(1-b)v_1 + a(1-b)v_2 + (1-a)bv_3 + abv_4$.

By $Q_0 = [A_1, A_2, A_3, A_4] = [0,1] \otimes [0,1]$ we denote the reference square, where $A_1 = (0,0), A_2 = (1,0), A_3 = (1,1), A_4 = (0,1)$, then

$$\begin{aligned} f: \quad & Q_0 \to Q \\ & (a, b) \to (x, y, z) = f(u, v) \end{aligned}$$

is a one-to-one mapping from Q_0 onto Q. Therefore, we only need to discuss how to derive a GC^1 surface on the reference square Q_0. We subdivide Q_0 into four triangles by joining A_1A_3 and A_2A_4, respectively, and by A_0 we denote the intersection point of A_1A_3 and A_2A_4.

Similarly, by

$$P_m(v) = \sum_{i\in\Gamma_{\delta_m}} (b_i u_{m,i}^3 + 3 \sum_{i\neq j\in\Gamma_{\delta_m}} b_{i,j}u_{m,i}^2 u_{m,j}) + 6b_{\delta_m} u_{m,0}u_{m,m+1}u_{m,m+2}$$

we denote the cubic patch on the triangle $\delta_m = [A_0, A_m, A_{m+1}]$, $m = 1,2,3,4$, respectively, where $A_5 = A_1$, and $(u_{m,0}, u_{m,m+1}, u_{m,m+2})$ is the barycentric coordinates of $v = u_{m,0}A_0 + u_{m,m}A_m + u_{m,m+1}A_{m+1}$ with respect to δ_m, respectively.

According to equation (3), if there exist some positive constants $\{\alpha_i'', \beta_i''\}_{i=1}^4$ such that for $v \in \delta_i \cap \delta_{i+3}$,

$$D_{i,0}P_i(v) = \alpha_i'' D_{i,i+1}P_i(v) + \beta_i'' D_{i,i+3}P_{i+3}(v), \quad 1 \leq i \leq 4, \tag{11}$$

then P, $P|_{\delta_i} = P_i$, is a GC^1 surface on δ, where $P_{i+4} = P_i$ and $D_{i,j+4}f = D_{i,j}f$ if $j \geq 1$.

If we set $\alpha_i'' = \beta_i'' = \frac{1}{2}$, then equation (11) is equivalent to

$$\begin{aligned} b_{i,0} &= \frac{1}{2}(b_{i,i+1} + b_{i,i+3}), \\ b_{0,i} &= \frac{1}{2}(b_{\delta_i} + b_{\delta_{i+3}}) \quad 1 \leq i \leq 4. , \\ b_0 &= \frac{1}{4}(b_{0,1} + b_{0,2} + b_{0,3} + b_{0,4}), \end{aligned} \tag{12}$$

So we can construct a GC^1 surface on a triangle δ by using four cubic patches as follows

I. The control vectors $\{b_i, b_{i,i+1}, b_{i,i+2}, b_{\delta_i}\}_{i=1}^4$ are chosen arbitrarily.

II. The other control vectors are determined by equation (12).

4. GC^1 surfaces on a space mesh

In this section, we will discuss the constructing of GC^1 surfaces on space meshes as defined in Section 1 by cubic patches.

We assume that corresponding to each vertex $v_i \in \Delta$, a position vector b_i and a plane π_i are given, respectively. We will construct a GC^1 surface on Δ, P, say, such that $b_i \in P$ and such that π_i is the tangent plane of P at b_i.

By P_δ we denote the GC^1 surface on a triangle element or on a quadrangle element $\delta \in \Delta$ obtained in Section 2 or Section 3, respectively. Thus, a surface P, $P|_\delta = P_\delta$, on the space mesh is oblained. For $v_i \in \Delta$, it holds $b_i \in P$. Therefore, we only need to derive the conditions for P being a GC^1 surface and the conditions for π_i being the tangent plane of P at b_i.

For two adjacent surfaces P_{δ_1} and P_{δ_2}, without loss of generality, we assume that δ_1 and δ_2 are two triangle elements in Δ, $\delta_i = [v_1, v_2, v_2 + i]$ $(i = 1,2)$, $e = \delta_1 \cap \delta_2$, and that

I To get a regular GC^1 surface, the lengthes of vectors $b_{i,j} - b_i$ and $b_{j,i} - b_j$ should not be too long. It is better to chose $b_{i,j}$ and $b_{j,i}$ such that

$$||b_{i,j} - b_i||, \quad ||b_{j,i} - b_j|| < \frac{1}{2}||v_j - v_i||.$$

II If we take $\{b_{i,j}, b_{j,i}\}_{e_{i,j}\in\delta}$ as an element in $\mathbb{R}^{4N}$ (the reason that we can do so is that $\{b_{i,j} - b_i\}_j \in \pi_i$ and $\{b_{j,i} - b_j\}_j \in \pi_j$ are coplanar vectors, respectively), then

$$m(\bigcup_{i,j} E_{i,j}) = 0,$$

where N is the number of edges in Δ, mA is the measure of the collection A, and $E_{i,j} := \{X \in \mathbb{R}^{4N} : \ J_{i,j}(X) := J_{i,j} = 0, \ \ J_{i,j}$ is defined in (13)$\}$. So the constraints of the choosing of the control vectors $\{b_{i,j}, b_{j,i}\}_{e_{i,j}\in\delta}$ are very mild.

References

[1] R.E. Barnhill, *Surfaces in computer aided geometric design: a survey with new results*, CAGD **2**(1985), 1—17.

[2] W. Boehm, *Cubic B-spline surves and surfaces in computer aided geometric design*, Computing **19**(1977), 29—34.

[3] W. Boehm, *cubics: a survey*, Computer Graphics and Image Processing, **19**(1982), 201–226.

[4] R. Clough and J.Tocher, *Finite elslent stiffness matrices for analysis of plates in blending*, In Proceedings of Conference on Matrix Methods in Structural Analysis, 1965.

[5] W. Dahmen, *Smooth Piecewise Quadric Surfaces*, Mathematical Methods in Computer Aided Geometric Design, Tom Lyche and L. Schümaker (eds.), 181—193.

[6] G. Farin, *Bezier Polynomials over triangles and the construction of piecewise C^r Polynomials*, Technical Report TR/91 Dept. Mathematics, Brunel Univ. Uxbridge, England, 1980.

[7] G. Farin, *Smooth interpolation top scattered 3-D date*, In R. Barnhill and W. Boehm, editors, Surfaces in Computer Aided Geometric Design, North-Holland, 1982.

[8] G. Farin (ed.), *Geometric Modeling: Algorithms and New Trends*, SIAM, 1987.

[9] D.Y. Liu, *A geometric condition for smoothness between adjacent Bezier surface patches*, ACTA Math. Appl. Sinica, Vol.9, No.4, Oct., 1986.

[10] B.Piper, *Visually smooth interpolation with triangular Bezier patches*, In G.Farin, editor, Geometric Modeling: Algorithms and New Trends, SIAM, 1987, 221—234.

[11] X.Q. Shi and T.J. Wang, *The GC^1 quintic interpolant on arbitrary triangular mesh*, J. of Mathematical Research & Exposition, **4**(1992), 583–588.

空间网格上的三次GC'插值格式

施 锡 泉　　　　　　王 天 军

(大连理工大学数学科学研究所，116024)　(北京航空航天大学)

摘　要

本文得到了由三次参数多项式构造的GC'插值格式，该插值格式定义在由空间三角形和空间四边形构成的空间网格上，并通过该网格的所有网点，同时在每个网点处以事先给定的平面为切平面.