projection, that is, $\|R\| = 1$. This is in constradiction with the definition of Y. So Y is not coproximinal. □

Theorem 3.4 *Let X be a Banach space. Then the following statements are equivalent.*

(1) *For each 1-dimensional subspace Y of X, the linear selection of R_Y is unique.*

(2) *X is smooth.*

Proof (1) ⇒ (2). For any $x_0 \in X$ and $\|x_0\| = 1$, let $Y = [x_0]$. By Theorem 2.4, R_Y has a linear selection R. Let $M = \ker R$. By Theorem 2.1, P_M has a linear selection P and $Px_0 = 0$. By a Theorem of [2], there exists a $f \in X_*$ such that $f(x_0) = \|f\| = \|x_0\| = 1$ and, for each $y \in M$, $f(y) = 0$. So f is a peak functional. Suppose f_0 is a peak functional of x_0 also. By the proof of Theorem 2.4, there exists a linear selection f_0 such that $\ker f_0 = \ker R_0$. By condition, we have $R = R_0$. So $\ker f = \ker f_0$ .Hence there exists a number α such that $|\alpha| = 1$ and $f = \alpha f_0$. It follows $\alpha = 1$ from $1 = f(x_0) = f_0(x_0) = \alpha$. So X is smooth.

(2) ⇒ (1). Suppose Y is an 1-dimensional subspace of X such that R_Y has linear selection R_i (i=1,2). Let $Y = [x_0]$ where $\|x_0\| = 1$. By the proof of (1) ⇒ (2), there exist peak functionals f_i of x_0 such that $\ker f_i = \ker R_i$. Since X is smooth, $f_1 = f_2$. So $\ker R_1 = \ker R_2$. Hence $R_2 = R_2$. □

References

[1] J. Lindenstrauss & L. Tzafriri, *Classical Banach spaces*, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[2] I. Singer, *Beat approximation in normed linear spaces by elements of linear subspaces*, Springer-Verlag, Berlin, Heidelberg, New York, 1970.

[3] F. Deutsch, *Linear selections for the metric projections*, J. Func. Anal., **49** (1982), 269-292.

[4] S. Elumalai, *Best coapproximation in normed linear spaces*, Methods of Functional Analysis in Approximation Theory (C.A. Micchelli, D.V. Pai & B.V. Limaye, eds.) ISNM, **76** (1986), Birkhauser Verlag, Basel, pp.165–172.

[5] G.S. Rao, *Best coapproximation in a normed linear space*, in "Approximation Theory 5" (e.d., C.K. Chui, etc.), pp.535–538, Acadmic press, New York, 1986.

[6] G.S. Rao & K.R. Chandrasekaran, *The modulas of continuity of the set-valued cometric projection*, Methods of Functional Analysis in Approximation Theory (C.A. Micchelli, D.V. Pai & B.V. Limaye, eds.) ISNM, **76** (1986), Birkhauser Verlag, Basel, pp.157–163.

关于余逼近的一点注记

宋 文 华

(大连理工大学数学科学研究所, 116024)

摘 要

本文证明了1- 维子空间和余1- 维子空间是余可逼近的, 且余度量射影有线性的选择, 并举出例子对任何维数不小于2 的有限维子空间未必是可余逼近的.

Journal of Mathematical Research & Exposition
Vol.14, No.4, Nov. 1994

A Remark on Coapproximation *

Song Wenhua
(Inst. of Math. Science, Dalian Univ. of Tech., Dalian 116024)

Abstrict In this paper, we show that the cometric projection R_Y admits a linear selection when Y is 1-dimensional or 1-codimensional and a finite dimensional subspace of a Banach space may be non-coproximinal.

Keywords coxpproxmntron, cometric projectron, coChebyshev.

1. Introduction

Let Y be a subspace of a normed linear space X. The cometric (resp. metric) projection onto Y is the set-valued mapping $R_Y : X \mapsto 2^Y$ (resp. $P_Y : X \mapsto 2^Y$) defined by, for each $x \in X$,

$$R_Y(x) = \{y \in Y; \quad \text{for every} \quad g \in Y, \quad \text{and} \quad \|g - y\| \leq \|x - g\|\}$$

(resp.

$$P_Y(x) = \{y \in Y; \quad \text{for every} \quad g \in Y, \|x - y\| \leq \|x - g\|\}.)$$

Y is called coproximinal (resp. coChebyshev) if $R_Y(x)$ contains at least (resp. exactly) one point for each $x \in X$. Y is called proximinal (resp. Chebyshev) if $P_Y(x)$ contains at least (resp. exactly) one point for each $x \in X$.

Suppose $F_Y : X \mapsto 2^Y$ is a set-valued mapping. A linear selection for F_Y is a linear map $F : X \mapsto Y$ such that $F(x) \in F_Y(x)$ for every $x \in X$. It is easy to verify that every linear selection for P_Y is a projection operator and the linear selection for R_Y is a constractive projection operator in usual sense.

The kernel of R_Y (resp. R_Y) is the set defined by $\ker R_Y = \{x \in X; \ \in R_Y(x)\}$. (resp. $\ker P_Y = \{x \in X; \ 0 \in P(x)\}$).

In [5], when Y is a coChebyshev subspace, G.S.Rao has put in that R_Y is linear if and only if $\ker R_Y$ contains a subspace N of X such that $X = Y \bigoplus N$.

The purpose of this paper is to consider the linear selection for R_Y when Y is 1-dimensional or 1-codimensional.

In section 3, we give an application of the Theorem 2.4. We prove that, for any integer $n \geq 2$, there exist a Banach space X and an n-dimensional subspace Y of X such that Y is non-coproximinal. It is well known that, in best approximation, if Y is a finite dimensional

*Received Aug. 10, 1992. Supported by the Ntional Science Foundation of P.R. China.

subspace then Y is proximinal. This fact can not be extened in best coapproximation.

2. Linear selections

The following Theorem expresses a relation between the best approximation and best coapproximation.

Theorem 2.1 *Let X be a Banach space. Suppose Y is a coproximinal subspace and R_Y admits a linear selection R. Then $M = \{x \in X;\ \ Rx = 0\}$ is proximinal and $P = id - R$ is a linear selection of P_M where id is the identity operator on X.*

If M is a proximinal subspace of X and P_M admits a linear selection P, then $Y = \{x \in X;\ \ Px = 0\}$ is coproximinal and $R = id - P$ is a linear selection of R_Y.

Proof Suppose $x \in X$. For any $y \in M$, since $Ry = 0$, for each $g \in Y$, $||g|| \leq ||y - g||$. Let $y_0 = y - Px$ and $g = x - Px = Rx$. Then $y_0 \in M$ and $g \in Y$ and

$$||x - Px|| \leq ||y_0 - (x - Px)|| = ||x - y||,$$

i.e. $Px \in P_Y(x)$. So M is proximinal and P is a linear selection of P_M.

Suppose P is a linear selection of P_M and $Y = \{x \in X;\ \ Px = 0\}$. Let $R = id - P$. We need only to show that $Rx \in R_Y(x)$ for each $x \in X$. For each $g\ inY$, since $Pg = 0$, for any $y \in M$, $||g|| \leq ||g - y||$. Let $g_0 = g - Rx = g - x - Px$ and $y = -Px$. Then $g_0 \in Y$ and $y \in M$. Thus

$$||g - Rx|| = ||g_0|| \leq ||g_0 - y|| = ||g - x|| = ||x - g||.$$

So Y is coproximinal and R is a linear selection. □

Theorem 2.2 *Let Y be a subspace of a linear normed space X and Y be coproximinal. Then R_Y admits a linear selection if and only if there exists a closed subspace $N \subseteq \ker R_Y$ such that $X = Y \bigoplus N$.*

Remark When Y is coChebyshev subspae, G.S.Rao [5] has proved the similar result.

Proof $(\Rightarrow)$. Let R be a linear selection and

$$N = \ker R = \{x \in X;\ \ Rx = 0\}.$$

It is obvious that $N \subseteq \ker R_Y$. Since R is a projection operator and $Y = R(R)$, we have $X = Y \bigoplus N$.

$(\Leftarrow)$. If there exists a subspace $N \subseteq \ker R_Y$ such that $X = Y \bigoplus N$, let $R(y + n) = y$ where $y \in Y$ and $n \in N$. Then R is a projection operator. It is enough to show $R(x) \in R_Y(x)$ for each $x \in X$.. Suppose $x = y + n$. Since $0 \in R_Y(n)$, for any $g \in Y$, $||g|| \leq ||n - g||$. So, for any $g \in Y$,

$$||g - y|| \leq ||n - (g - y)|| = ||x - g||,$$

that is, $y \in R_Y(x)$. □

This Theorem is similar to the case in best approximation, which was proved by F.Deutsch [3]. In words, R_Y has a linear selection if and only if Y has a complement in

$\ker R_Y$.

It is similar to the case in best approximation, we have the following Proposition.

Proposition Let Y be a coproximinal subspace. Then $\ker R_Y$ is a subspace if and only if Y is coChebyshev and R_Y is linear.

As applications of Theorem 2.2,, we have the following Theorems.

Theorem 2.4 *If Y is a coproximinal hyperplane of a Banach space X, then R_Y admits a linear selection.*

Proof Let $x \in X \setminus Y$. Suppose $g_0 \in R_Y(x)$. Let $x_0 = x - g_0$. It is obvious that $[x_0] \bigoplus Y = X$. For any $g \in Y$,

$$\|g\| = \|(g + g_0) - g_0\| \leq \|x - (g + g_0\| = \|(x - g_0) - g\|.$$

So $0 \in R_Y(x_0)$. It is evident that

$$[x_0] \subseteq \{x \in X;\ \ 0 \in R(x)\}.$$

By theorem 2.2, R_Y admits a linear selection. □

Theorem 2.5 *If Y be an 1-dimensional subspace of a Banach space X, then Y is coproximinal and R_Y admits a linear selection.*

Remark We shall see, in section 3, that, for any integer $n \geq 2$, there exist a Banach space X and an n-dimensional subspace Y of X such that Y is non-coproximinal.

Proof Let $x_0 \in X$ such that $\|x_0\| = 1$ and $Y = [x_0]$. Suppose f is a peak functional of x_0. Let $M = \{x \in X;\ \ f(x) = 0\}$. For each $x \in X$, then $y_0 = x - f(x)x_0 \in Y$.

For any $y \in M$,

$$\|x - y_0\| = |f(x)| = |f(x - y)| \leq \|x - y\|.$$

So $y_0 \in P_M(x)$. It is obvious that $Px = x - f(x)x_0$ is linear. So P is a linear selection of P_M. By Theorem 2.1 and 2.2, the $Y = \{x \in X;\ \ Px = 0\} = [x_0]$ is coproximinal and $R = id - P$ is a linear selection of R_Y. □

3. The non-existence of best co-approximation and the uniqueness of linear selections.

We will need to use the following Theorem.

Theorem 3.1 (J.Lindenstrauss & L.Tzafriri, [1]) *Let $X = l_p$, for some $1 < p < \infty, p \neq 2$ and let P be a projection of norm 1 in X. Then there exist vector $\{u_i\}_{i=1}^m$ of norm 1 in X (where $m = \dim R(P)$ is either an integer or ∞) which satisfies $\mathrm{supp}(u_i) \cap \mathrm{supp}(u_j) = \emptyset$ so that $Px = \sum_{i=1}^m u_i^*(x)u_i$, where $\{u_i^*\} \subseteq X^*$ satisfy*

$$\|u_i^*\| = u_i^*(u_i) = 1,\ \ i = 1, 2, \cdots, m.$$

It is well known that, in best approximation, if Y is a finite dimensional subspace of a linear normed space X, then Y is proximinal. However, this fact is not truth in best coapproximation. In fact, we have the following propositions.

For any set A, $cardA$ will denote the cardinality of A. Also, if x is a function defined on some set T, the support of x is the set

$$\mathrm{supp}(x) = \{t \in T;\ \ x(t) \neq 0\}.$$

Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of Banach spaces and $1 \le p < \infty$, we denote by $(\bigoplus_{\lambda\in\Lambda} X_\lambda)_p$ the space

$$\{f : \Lambda \mapsto \bigcup_{\lambda\in\Lambda} X_\lambda;\ \ \sum \|f(\lambda)\|^p < \infty\}$$

endowed the norm

$$\|f\| = [\sum_{\lambda\in\Lambda} \|f(\lambda)\|^p]^{1/p}.$$

Proposition 3.2 For any $n \ge 3$, there exist an n-dimensional Banach space X and a hyperplane Y of X such that, for any projection P, if $R(P) = Y$, then $\|P\| > 1$.

Proof Fix a p, $1 < p < \infty$ and $p \neq 2$. Let $X = l_p(n)$ and

$$Y = \{x \in X;\ \ \sum_{k=1}^{n} x(k) = 0\}.$$

Then codim$Y = 1$. If there exists a constractive projection P such that $R(P) = Y$. Define the projection Q as, for any $x \in l_p$, $(Qx)(k) = 0$ when $k > n$ and $(Qx)(k) = x(k)$ when $k \le n$. It is evident that PQ is a constractive projection defined on l_p and $R(PQ) = Y$. By Theorem 3.1, there exist $\{x_i\} \subseteq X$ of norm 1 such that $\mathrm{supp}(x_i) \cap \mathrm{supp}(x_j) = \emptyset$ and $Y = (\bigoplus_{k=1}^{n-1}[x_i])_p$ where $[x] = \mathrm{span}\{x\}$. If there exists an i such that $\mathrm{card}[\mathrm{supp}(x_i)] = 1$, assume $\mathrm{supp}(x_i) = k_0$. Then $x_i(k_0) \neq 0$ and $x_i(j) = 0$ for each $j \neq k_0$. But $x_i \in Y$, by definition, $0 = \sum_{k=1}^{n} x_i(k) = x_i(k_0)$. This is in constradiction with $x(k) \neq 0$. So we get $\mathrm{card}[\mathrm{supp}(x_i)] \ge 2$ for any i. Since the sets $\mathrm{supp}(x_i)$ are disjoint and $n \ge 3$, we get

$$\bigcup_{k=1}^{n-1} \mathrm{supp}(x_k) = \mathrm{supp}(\sum x),$$

and

$$\mathrm{card}[\bigcup_{k=1}^{n-1} \mathrm{supp}(x_k)] \ge 2(n-1) > n.$$

This is in constradiction with the $\sum_{k=1}^{n-1} x_k \in l_p(n)$. Thus, for any projection P, if $R(P) = Y$, then $\|P\| > 1$. □

Proposition 3.3 For each integer $n \ge 2$, there exist a Banach space X and an n-dimensional subspace Y such that Y is non-coproximinal.

Proof. By the proposition 3.2, there exist an $n+1$-dimensional Banach space X and a hyperplane Y of X such that , for any projection P, if $R(P) = Y$, then $\|P\| > 1$. By Theorem 2.4, if Y is coproximinal,then R_Y has a linear selection R and R is a constractive projection, that is, $\|R\| = 1$. This is in constradiction with the definition of Y. So Y is not coproximinal. □

Theorem 3.4 *Let X be a Banach space. Then the following statements are equivalent.*

(1) *For each 1-dimensional subspace Y of X, the linear selection of R_Y is unique.*

(2) *X is smooth.*

Proof (1) $\Rightarrow$ (2). For any $x_0 \in X$ and $\|x_0\| = 1$, let $Y = [x_0]$. By Theorem 2.4, R_Y has a linear selection R. Let $M = \ker R$. By Theorem 2.1, P_M has a linear selection P and $Px_0 = 0$. By a Theorem of [2], there exists a $f \in X_*$ such that $f(x_0) = \|f\| = \|x_0\| = 1$ and, for each $y \in M$, $f(y) = 0$. So f is a peak functional. Suppose f_0 is a peak functional of x_0 also. By the proof of Theorem 2.4, there exists a linear selection f_0 such that $\ker f_0 = \ker R_0$. By condition, we have $R = R_0$. So $\ker f = \ker f_0$ .Hence there exists a number α such that $|\alpha| = 1$ and $f = \alpha f_0$. It follows $\alpha = 1$ from $1 = f(x_0) = f_0(x_0) = \alpha$. So X is smooth.

(2) $\Rightarrow$ (1). Suppose Y is an 1-dimensional subspace of X such that R_Y has linear selection R_i (i=1,2). Let $Y = [x_0]$ where $\|x_0\| = 1$. By the proof of (1) $\Rightarrow$ (2), there exist peak functionals f_i of x_0 such that $\ker f_i = \ker R_i$. Since X is smooth, $f_1 = f_2$. So $\ker R_1 = \ker R_2$. Hence $R_2 = R_2$. □

References

[1] J. Lindenstrauss & L. Tzafriri, *Classical Banach spaces*, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[2] I. Singer, *Beat approximation in normed linear spaces by elements of linear subspaces*, Springer-Verlag, Berlin, Heidelberg, New York, 1970.

[3] F. Deutsch, *Linear selections for the metric projections*, J. Func. Anal., **49** (1982), 269-292.

[4] S. Elumalai, *Best coapproximation in normed linear spaces*, Methods of Functional Analysis in Approximation Theory (C.A. Micchelli, D.V. Pai & B.V. Limaye, eds.) ISNM, **76** (1986), Birkhauser Verlag, Basel, pp.165–172.

[5] G.S. Rao, *Best coapproximation in a normed linear space*, in "Approximation Theory 5" (e.d., C.K. Chui, etc.), pp.535–538, Acadmic press, New York, 1986.

[6] G.S. Rao & K.R. Chandrasekaran, *The modulas of continuity of the set-valued cometric projection*, Methods of Functional Analysis in Approximation Theory (C.A. Micchelli, D.V. Pai & B.V. Limaye, eds.) ISNM, **76** (1986), Birkhauser Verlag, Basel, pp.157–163.

关于余逼近的一点注记

宋文华

(大连理工大学数学科学研究所，116024)

摘 要

本文证明了1- 维子空间和余1- 维子空间是余可逼近的，且余度量射影有线性的选择，并举出例子对任何维数不小于2 的有限维子空间未必是可余逼近的.