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Uniform Stability of Certain Nonlinear Systems of Neutral Functional Differential Equations with Infinite Delay *

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Abstract In this paper, we investigate the asymptotic behavior of the solutions and uniform stability of the zero solutions of certain nonlinear systems of neutral functional differential equations with infinite delay and obtain some simple criteria for stability.

Keywords neutral system, infinite delay, uniform stability

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By $BC(I,D)$ we denote the set of all the bounded continuous functions from I to D, by $\|\cdot\|$ we denote a certain norm in R^n.

Consider nonlinear neutral system

$$\dot{x}(t)=F(t,x_t,\dot{x}(t-r(t)),x(t-r_1(t)),\cdots,x(t-r_m(t))), \tag{1}$$

where $x\in R^n, 0\le r(t),r_1(t),\cdots,r_m(t)\le t, t\in R^+, x_t$ is defined as $x_t(\theta)=x(t+\theta)$ for $-\sigma(t)\le\theta\le 0, \sigma(t)=\max[r(t),r_1(t),\cdots,r_m(t)], F(t,\cdot,\cdots,\cdot)\in C(R^+\times BC(R^-,R^n)\times R^n\times\cdots\times R^n,R^n), F(t,0,0,0,\cdots,0)=0$ for any $t\in R^+$. We assume that

(H$_1$) $\|F(t,x_t,y,x_1,\cdots,x_m)\|\le I(t,|x_t|,\|y\|,\|x_1\|,\cdots,\|x_m\|)$

for

$$(t,x_t,y,x_1,\cdots,x_m)\in R^+\times BC(R^-,R^n)\times R^n\times R^n\times\cdots\times R^n,$$

where $|x_t|=\sup_{-\sigma(t)\le\theta\le 0}\|x(t+\theta)\|, I(t,\xi,\eta,\xi_1,\cdots,\xi_m)\in C(R^+\times R^+\times R^+\times R^+\times\cdots\times R^+,R^+), I(t,\xi,\eta,\xi_1,\cdots,\xi_m)$ is monotone nondecreasing in $(\xi,\xi_1,\cdots,\xi_m), I(t,0,0,0,\cdots,0)=0$ for $t\in R^+$;

(H$_2$) For any monotone nondecreasing function $v(t)\in C(R,R^+)$ with $\|x(\xi)\|\le v(t), t-\sigma(t)\le\xi\le t$, the inequality

$$\|\dot{x}(t)\|\le I(t,v(t),\|\dot{x}(t-r(t))\|,v(t),\cdots,v(t))$$

implies

$$\|\dot{x}(t-r(t))\|,\|\dot{x}(t)\|\le J(t,v(t)),$$

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where $J(t,v) \in C(R^+ \times R^+, R^+), J(t,0) = 0, J(t,v)$ is monotone nondecreasing in v (cf. [1,2,3] etc.);

(H_3) For initial conditions

$$x(t) = \phi(t), \dot{x}(t) = \dot{\phi}(t), -\infty < t \le t_0 \quad (t_0 \in R^+)$$

there exists a unique solution $x(t) = x(t, t_0, \phi)$ to (1), where $\phi(t) \in BC((-\infty, t_0], R^n), \dot{\phi}(t) \in C((-\infty, t_0], R^n)$;

(H_4) For any $t_0, w_0 \in R^+$, there exists a unique solution $w(t) = w(t, t_0, w_0)$ defined on $[t_0, +\infty)$ for the initial value problem

$$\begin{cases} \dot{w}(t) = I(t, w(t), J(t, w(t)), w(t), \cdots, w(t)) \\ w(t_0) = w_0. \end{cases} \tag{2}$$

Theorem 1 (Comparison Theorem) *If the Conditions (H_1)-(H_4) are satisfied, then*

$$\|x(t, t_0, \phi)\| \le w(t, t_0, |\phi|), t \in [t_0, +\infty), \tag{3}$$

where $|\phi| = \sup_{-\infty < t \le t_0} \|\phi(t)\|$.

Proof Define

$$v(t) = \begin{cases} |\phi|, t \in (-\infty, t_0] \\ |\phi| + \int_{t_0}^{t} \|\dot{x}(s)\| ds, t \in [t_0, +\infty). \end{cases} \tag{4}$$

Then $v(\cdot)$ is monotone nondecreasing on R. By the conditions one gets

$$\dot{v}(t) = \|\dot{x}(t)\| \le I(t, v(t), J(t, v(t)), v(t), \cdots, v(t)), t \in [t_0, +\infty) \tag{5}$$

this induces the result.

By Theorem 1 we directly have the following

Theorem 2 *If (H_1)-(H_4) are satisfied and te zero solution of (2) is uniformly stable with respect to all the positive solutions, then the zero solution of (1) is also uniformly stable.*

Remark 1 In Th.1-4, $I(t, v, J(t,v), v, \cdots, v)$ can be replaced by $J(t,v)$.

Remark 2 Condition (H_2) is weaker than that in [1,2,3], it can be realized for many equations, which can be shown by the example below.

Remark 3 Under the conditions of Th.3, the zero solution of (1) is stable in the $C^{(1)}$-Space ([4]).

Theorem 3 *Assume (H_1)-(H_3) are satisfied and*

$$I(t, \xi, J(t,\xi), \xi, \cdots, \xi) = I^*(t) I^{**}(\xi),$$

where $I^{**}(\cdot)$ is Lipschitzian and monotone nondecreasing, $I^{**}(0) = 0, I^{**}(\xi) > 0$ for $\xi > 0$. If

$$\int^{+\infty} [I^{**}(\xi)]^{-1} d\xi = +\infty, \quad \int^{+\infty} I^*(s) ds < +\infty, \tag{6}$$

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then the zero solution of (1) is uniformly stable and stable in the $C^{(1)}$- Space.

Proof We first point out that (H_4) is satisfied under (6). Thus, the remainder we need only to prove that the zero solution of (2) is uniformly stable with respect to all the positive solutions.

Define

$$\Phi(\omega) = \int_{\frac{\epsilon}{2}}^{\omega} [I^{**}(\xi)]^{-1} d\xi, \quad \omega \in R^{+}$$

for given $\epsilon > 0$. It is obvious that $\Phi^{-1}(\cdot)$ is well-defined and is monotone increasing because $\Phi'(\omega) > 0$. Let

$$\delta = \frac{1}{2}\min[\frac{\epsilon}{2}, \Phi^{-1}(\frac{1}{2}\Phi(\epsilon))].$$

By (6), there exists a certain constant $T > 0$ such that

$$\int_{T}^{+\infty} I^{*}(s)ds < \frac{1}{2}\Phi(\epsilon). \tag{7}$$

I. Suppose t_0, ω_0 are given, $t_0 \geq T, \omega_0 \in R^{+}, 0 < \omega_0 < \delta, \omega(t) = \omega(t, t_0, \omega_0)$. It is easy to show that

$$\omega(t) = \Phi^{-1}(\Phi(\omega_0) + \int_{t_0}^{t} I^{*}(s)ds).$$

By the estimation

$$\Phi(\omega_0) < \Phi(\delta) < \Phi(2\delta) < \frac{1}{2}\Phi(\epsilon),$$
$$\int_{t_0}^{t} I^{*}(s)ds < \int_{T}^{+\infty} I^{*}(s)ds < \frac{1}{2}\Phi(\epsilon) \quad \text{for} \quad t \geq t_0 (\geq T),$$

we have

$$\omega(t) < \Phi^{-1}(\frac{1}{2}\Phi(\epsilon) + \frac{1}{2}\Phi(\epsilon)) = \epsilon \quad \text{for} \quad t \in [t_0, +\infty). \tag{8}$$

II. By the continuous dependence of solutions on initial conditions, for any $0 \leq \tau \leq T$, there exists a constant $\delta(\tau, \epsilon), 0 < \delta(\tau, \epsilon) < \epsilon$, such that

$$\omega(t, t_0, \omega_0) < \delta \quad \text{for} \quad t \in [t_0, T]$$

if $0 < \omega_0 < \delta(\tau, \epsilon), t_0 \in (\tau - \delta(\tau, \epsilon), \tau + \delta(\tau, \epsilon)) \bigcap R^{+}$. Now $[0, T] \subset \bigcup_{0 \leq \tau \leq T}((\tau - \delta(\tau, \epsilon), \tau + \delta(\tau, \epsilon)) \bigcap R^{+})$, so there exists a finite set of points $\{\tau_i | 0 \leq \tau_i \leq T, i = 1, 2, \cdots, k\}$ such that $[0, \tau] \subset \bigcup_{1 \leq i \leq k}((\tau_i - \delta(\tau_i, \epsilon), \tau_i + \delta(\tau_i, \epsilon)) \bigcap R^{+})$. Let

$$\delta' = \min[\delta, \delta(\tau_i, \epsilon), i = 1, 2, \ , k],$$

then δ' depends only on ϵ, and $\delta' \leq \delta < \epsilon$.

By the meaning of δ', it is easy to show that, if $0 \leq t_0 \leq T, 0 < \omega_0 < \delta'$, then

$$\omega(t, t_0, \omega_0) < \delta \quad \text{for} \quad t \in [t_0, T].$$

III. Let $t_0 \in R^{+}, 0 < \omega_0 < \delta', \omega(t) = \omega(t, t_0, \omega_0)$.

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(i) If $t_0 \geq T$, then (8) holds;
(ii) If $0 \leq t_0 \leq T$, then

$$0 < \omega(t, t_0, \omega_0) < \delta < \frac{1}{2}\epsilon < \epsilon \quad \text{for} \quad t \in [t_0, T]$$

thus $0 < \omega(T, t_0, \omega_0) < \delta$ and

$$0 < \omega(t, t_0, \omega_0) = \omega(t, T, \omega(T, t_0, \omega_0)) < \epsilon, t \geq T$$

by (8). Therefore we have

$$0 < \omega(t, t_0, \omega_0) < \epsilon \quad \text{for} \quad t \in [t_0, +\infty).$$

This completes the proof.

Theorem 4 *Under the conditions of Theorem 3, each solution of (1) tends to a constant as $t \to +\infty$.*

Proof One can show that the solution of (1) is uniformly bounded, thus, by (5) one easily show $\|\dot{x}(t)\| \in L^1[0, +\infty)$, and the conclusion follows from [4,Lemma 2.4].

A simple new criterion for the retarded equation

$$\dot{x}(t) = F(t, x_t) \tag{9}$$

can be obtained by Th.3.

Corollary 1 *If $\|F(t, \phi)\| \leq I^*(t)I^{**}(|\phi|)$ for $(t, \phi) \in R^+ \times BC(R^-, R^n)$, where $I^*(\cdot)$ and $I^{**}(\cdot)$ are as in Th.3, then the zero solution of (9) is uniformly stable. Furthermore, each solution of (9) tends to a constant as $t \to +\infty$.*

Example Consider the nonlinear neutral scalar equation

$$\dot{x}(t) = \frac{1}{(t+1)^2}[x(t) + \sin x(t - [t])] - e^{-t-[\dot{x}(t-1)]^2} x([t]). \tag{10}$$

We have

$$x(t) = e^{-\frac{1}{t+1} + \frac{1}{n+1}} x(n) + e^{-\frac{1}{t+1}} \int_n^t e^{\frac{1}{s+1}} [\frac{1}{(s+1)^2} \sin x(s - [s]) - e^{-s-[\dot{x}(s-1)]^2} x([s])] ds$$
$$(n \leq t \leq n+1, [s] = n).$$

By this equality one easily proves the global existence of solutions of (10). Besides, from (10) we have

$$|\dot{x}(t)| \leq I^*(t)(|x(t - [t])| + |x([t])| + |x(t)|) \leq I^*(t)I^{**}(|x_t|),$$

where $I^*(t) = \max[(t+1)^{-2}, e^{-t}]$, $I^{**}(\xi) = 3\xi$. Obviously,

$$\int^{+\infty} [I^{**}(\xi)]^{-1} d\xi = +\infty, \quad \int^{+\infty} I^*(s) ds < +\infty,$$

thus, by Th.3,4, the zero solution of (10) is uniformly stable and each solution of (10) converges at infinity.

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一类非线性无穷时滞中立型泛函微分方程系统的一致稳定性

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摘　要

本文讨论一类无穷时滞非线性中立型泛函微分方程解的渐近性态与零解的一致稳定性，得到若干简单的稳定性判据.

关键词　中立型系统，无穷时滞，一致稳定性.

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