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On The Extensions of Integral Representations in C^n *

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Abstract We point out paper [4] does not get anything new, but it is only the special case of Leray formula and Cauchy-Fantappié formula.

Keywords Cauchuy-Fantappié formula, Leray formula, special case.

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There are many very good results in the theory of integral representations in C^n. The most important formulae are Cauchy-Fantappié formula[1], Leray formula[2] and Koppelman-Leray formula[2] of exterior differential forms of type $(0,q)$. The common characterization of the above three formulae is that they are contructed in bounded domains in C^n. Therefore, these three formulae have generality.

Leray contructed the Cauchy-Fantappié formula from the viewpoint of homology theory in 1959. We now state the Cauchy-Fantappié formula as the following:

Let D be a bounded domain in $C^n, \phi(z)$ be a holomorphic function in D and a continuous function on $\bar{D}$, and $\beta \in h$ (some homology class) be a circulation. Then

$$\phi(z) = \frac{(n-1)!}{(2\pi i)^n} \int_\beta \phi(\xi) \sum_{k=1}^{n} (-1)^{k-1} u_k du_{[k]} \wedge d\xi.$$

This formula is the generalization of the regeneralation formula for holomorphic functions in the bounded domain of C^n. All the integral representation formulae for holomorphic functions can be reduced from the Cauchy-Fantappié formula. The reason is that the circulation β in the Cauchy-Fantappié formula has a lot of choices. From the view point of solving problems, if we only want to reproduce differentiable functions in bounded domains, then Bochner-Martinelli formula $u(z) = B_{\partial D}u(\xi) - B_D\bar{\partial}u(\xi)$ is very good. But, if we want to obtain the solution to the equation $\bar{\partial}v = f$ in the bounded domain of C^n, then the Bochner-Martinelli formula does not work, since the Bochner-Martinelli kernel is not holomorphic with respect to z. For the reason mentioned above, we introduce a parameter λ and set

$$\eta_j = \lambda \frac{\bar{\xi}_j - \bar{z}_j}{|\xi - z|^2} + (1-\lambda)u_j,$$

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where $u=(u_1,\cdots,u_n)$ and $\langle u,\xi-z\rangle=1,(\xi,z)\in\partial D\times D$ or $u_j=\tilde{w}_j/\langle\tilde{w},\xi-z\rangle,\langle\tilde{w},\xi-z\rangle\neq 0$, and $(\xi,z)\in\partial D\times D$. We have Leray formula: $u=L^{\tilde{w}}_{\partial D}u-R^{\tilde{w}}_{\partial D}\bar{\partial}u+B_D\bar{\partial}u$. Therefore, for any given bounded domain of C^n, we choose $u_j(\xi,z)$ with $u_j(\xi,z)$ is holomorphic with respect to z (for example, in strictly pseudoconvex domains), then $v=-R^{\tilde{w}}_{\partial D}f-B_Df$ is the solution to the equation $\bar{\partial}v=f$. We must note that, as in the Cauchy-Fanteppié formula, the u_j in the Leray formula has a lot of choices. In fact, any differentiable vector-valued function $\tilde{w}$ with $\langle\tilde{w},\xi-z\rangle\neq 0$ can be used as $\tilde{w}$ in Leray formula (if necessary we choose some real parameter λ_j). Therefore, Leray formula is already the most generalized form of integral representation formulae (in this case the $\tilde{w}$ is defined on the whole boundary ∂D).

Let $w=(w_1,\cdots,w_n)$, where $w_j(\xi,z)\in C^1(\partial D\times D),(j=1,2,\cdots,n)$, and $\langle w,\xi-z\rangle\neq 0$, and $(\xi,z)\in\partial D\times D$. Then, we have the following expressions of Cauchy-Fantappié integral kernel:

$$K(\xi,z)=\frac{(n-1)!\omega'(w)\wedge d\xi}{(2\pi i)^n\langle w,\xi-z\rangle^n},$$

where $\omega'(w)=\sum_{j=1}^n(-1)^{j-1}w_jdw_{[j]}$, and $d\xi=d\xi_1\wedge\cdots\wedge d\xi_n$.

$$\begin{aligned}K(\xi,z)&=\frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi i)^n}\frac{\langle w,d\xi\rangle}{\langle w,\xi-z\rangle}\wedge d\frac{\langle w,d\xi\rangle}{\langle w,\xi-z\rangle}\wedge\cdots\wedge d\frac{\langle w,d\xi\rangle}{\langle w,\xi-z\rangle},\\&=\frac{1}{(2\pi i)^n}\frac{1}{\langle w,\xi-z\rangle^n}\det{}_{(n)}(w,\bar{\partial}_\xi w,\cdots,\bar{\partial}_\xi w)\wedge d\xi\\K(\xi,z)&=\frac{(n-1)!}{(2\pi i)^n}\sum_{j=1}^n(-1)^{j-1}u_jdu_{[j]}\wedge d\xi=\frac{1}{(2\pi i)^n}\det{}_{(n)}(u,\bar{\partial}_\xi u,\cdots,\bar{\partial}_\xi u),\end{aligned}$$

where $u_j=\frac{w_j}{\langle w,\xi-z\rangle}$. The integral kernels mentioned above have the following properties:

1. $d_\xi K(\xi,z)=0$;
2. for any given differentiable function ψ, the following is true

$$\frac{\omega'(\psi w)\wedge d\xi}{\langle\psi w,\xi-z\rangle^n}=\frac{\omega'(w)\wedge d\xi}{\langle w,\xi-z\rangle^n}.$$

Cauchy-Fantappié integral kernel has the following generalized expression[3]:

$$\begin{aligned}&\Omega(w^{(0)},w^{(1)},\cdots,w^{(n-1)},\xi-z)\\&=\frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi i)^n}\frac{\langle w^{(0)},d\xi\rangle}{\langle w^{(0)},\xi-z\rangle}\wedge d\frac{\langle w^{(1)},d\xi\rangle}{\langle w^{(1)},\xi-z\rangle}\wedge\cdots\wedge d\frac{\langle w^{(n-1)},d\xi\rangle}{\langle w^{(n-1)},\xi-z\rangle},\end{aligned}$$

where $w^{(i)}=(w_1^{(i)},\cdots,w_n^{(i)}),i=0,1,\cdots,n-1$, are vector- valued functions of the class C^1, and $\langle w^{(i)},\xi-z\rangle\neq 0,\xi\in\partial D,z\in D$. Specially, if $w^{(1)}=\cdots=w^{(n-1)}=w^{(0)}$, then

$$\Omega(w^{(0)},\cdots,w^{(0)},\xi-z)=\frac{(n-1)!}{(2\pi i)^n}\sum_{j=1}^n(-1)^{j-1}u_j^{(0)}du_{[j]}^{(0)}\wedge d\xi,$$

where $u_j^{(0)} = \dfrac{w_j^{(0)}}{\langle w^{(0)}, \xi - z\rangle}$. Cauchy- Fantappié integral kernel has the following improtant properties:

1. $\Omega(w^0, w^1, \cdots, w^{n-1}, \xi - z)$ does not depend on the vector- valued function w^0.

2. If $w^{(1)}, \cdots, w^{(n-1)}, v^{(1)}, \cdots, v^{(n-1)}$, are vector- valued functions of the class $C^2(\partial D)$, and vector-valued functions $w^{(0)}, v^{(0)}$ are in the class $C(\partial D)$, then the difference

$$\Omega(w^{(0)}, w^{(1)}, \cdots, w^{(n-1)}, \xi - z) - \Omega(v^{(0)}, v^{(1)}, \cdots, v^{(n-1)}, \xi - z)$$

is a $\bar{\partial}$-exact form. Specially, let $w^{(0)}$ and $\tilde{w}^{(0)}$ are two vector-valued functions of the class C^1, then

$$\begin{aligned} &\Omega(w^{(0)}, \cdots, w^{(0)}, \xi - z) - \Omega(\tilde{w}^{(0)}, \cdots, \tilde{w}^{(0)}, \xi - z) \\ &= \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} (-1)^{j-1} u_j^{(0)} du_{[j]}^{(0)} \wedge d\xi - \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} (-1)^{j-1} v_j^{(0)} dv_{[j]}^{(0)} \wedge d\xi \end{aligned}$$

is a $\bar{\partial}$-exact form, where $u_j^{(0)} = \dfrac{w_j^{(0)}}{\langle w^{(0)}, \xi - z\rangle}$, and $v_j^{(0)} = \dfrac{\tilde{w}_j^{(0)}}{\langle \tilde{w}^{(0)}, \xi - z\rangle}$. We can easily prove $R_{\partial D}^{\tilde{w}} \bar{\partial} u = L_{\partial D}^{\tilde{w}} u - B_{\partial D} u$. Therefore, $R_{\partial D}^{\tilde{w}^{(0)}} \bar{\partial} u = R_{\partial D}^{w^{(0)}} \bar{\partial} u$. From the above we can see that both Cauchy-Fantappié formula and Leray formula are independent of the choice of u if u stisfies $\langle u, \xi - z\rangle = 1$.

Let us take a look at the integral kernels in [4]. $H_j^{(m)}$ are nothing but

$$\begin{aligned} H_j^{(m)} &= T^{(m)} h_j = \lambda_m g_j^{(m)} + (1 - \lambda_m)[h_j + c_{m-1}(g_j^{(m-1)} - h_j) + \cdots + c_2(g_j^{(2)} - h_j)] \\ &= \lambda_m g_j^{(m)} + T^{(m-1)} h_j, \end{aligned}$$

where $c_2, \cdots, c_{m-1}$ consist of $\lambda_2, \cdots, \lambda_{m-1}$ and are constant, and

$$h_j = \frac{w_j}{\langle w, \xi - z\rangle}, g_j^{(m)} = \frac{w_j^{(m)}}{\langle w^{(m)}, \xi - z\rangle}, w_j^{(m)} = |\xi_j - z_j|^{m-2}(\bar{\xi}_j - \bar{z}_j).$$

Then author of [4] claims that the (32) in Theorem 2 in [4]

$$f(z) = \int_{\partial D} f(\xi) K(\lambda_2, \cdots, \lambda_p, \xi, z, w),$$

where $z \in D$, and $-\infty < \lambda_2, \cdots, \lambda_p < +\infty, p = 2, \cdots, N$, is an extension of Cauchy-Fantappié formula. It seems that the author of [4] misunderstands Cauchy-Fantappié formula, since if we compare (32) in [4] with Cauchy-Fantappié formula, we can see that (32) in [4] is not an extension of Cauchy-Fantappié formula, and in contrast it is one special case of Cauchy-Fantappié formula. In fact, (32) and (11) in [4] are the special cases of Cauchy-Fantappié formula if we let $u_j = T^{(m-1)} h_j = h_j + c_m(g_j^{(m-1)} - h_j) + \cdots + c_2(g_j^{(2)} - h_j)$, and $u_j = |\xi_j - z_j|^{m-2}(\bar{\xi}_j - \bar{z}_j)$ respectively. Theorem 1 in [4] is meaningless, too, since it directly follows from Leray formula or can be considered as one of corollaries of Leray fomula or an expression obtained by using some transforms. In fact, the author adds extra parameters in the integral kernels in (18) and (32) in [4] (except the last parameter λ_m which plays the same role as λ in Leray formula, all others have no essential use). The author of [4] reformulates Cauchy-Fantappié formula and Leray formula in complicated forms. It seems that from (18) and (32) in [4] the author could obtain the new results when m takes different values. But in fact, the expected results (for example, corollary 1-5 in [4]) are basicly independent of the choices of m, since no matter what m is, $T^{(m-1)}h_j$ are in the range of u in Leray formula and Cauchy- Fantappié formula (see paragraph 2 and 3). Therefore, from (18) and (32) in [4] we can not get the results beyond what we can obtain from Cauchy- Fantappié formula and Leray formula. (Note that from the (18) and (32) in [4] we can not get the integral representation formulae in the polydisc and in the analytic polyhedron by choosing parameters $\lambda_2, \cdots, \lambda_{m-1}$. Because of the same reasons mentioned above, the method use in the proof of the (18) in [4] is the same as the method use in the proof of Leray formula. In fact, if we let $\lambda = \lambda_m$ and $u_j = H_j^{(m-1)}$, and replace $\frac{\bar{\xi}_j - \bar{z}_j}{|\xi - z|^2}$ by $g_j^{(m)}$ in the vector-valued function $\eta^w = (\eta_1^w, \cdots, \eta_n^w)$ which used in the proof of Leray formula, where $\eta_j^w = (1-\lambda)u_j + \frac{\bar{\xi}_j - \bar{z}_j}{|\xi - z|^2}$, we can easily obtain the Theorem 1 in [4]. We must pointout that such modification is meaningless, since using $g_j^{(m)}$ to replace the $\frac{\bar{\xi}_j - \bar{z}}{|\xi - z|^2}$ in Leray formula does not agree with the principle of simplification, only increase the work of computation, and does not get anything new. It seems that the author of [4] does not quite understand why so many people use $\frac{\bar{\xi}_j - \bar{z}_j}{|\xi - z|^2}$ or Bochner-Martinelli formula as starting point. Therefore, the author thinks that the (18) and (32) in [4] are nothing but verifing Leray formula and Cauchy-Fantappié formula.

References

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关于 C^n 中积分表示的拓广问题

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摘 要

指出文[4] 所得的结果只是Leray 公式和Cauchy-Fantappié 公式的特殊情况.