Proof. (5.21) can be easily obtianed by the help of (5.5), (5.6), (5.8), (2.1), (2.2), (2.5), and (2.12). (5.22) follows from (5.21) and (2.7)–(2.10).

Theorem 3. *Let $f(x)$ be a continuous function satisfying the Zygmund condition*

$$|f(x+h)-2f(x)+f(x-h)|=o(h)$$

in $[-1,1]$ and let $|\beta_i|\le o(n)(1-x_i^2)^{-\frac{1}{2}}$, then the sequence of interpolatory polynomials $Q_{2n-1}(x)$ in (4.1) (with $\alpha_i=f(x_i)$) converges uniformly to $f(x)$ in $[-1,1]$.

Proof. The proof of this theorem could be obtianed on the same lines as in [4] and is omitted here.

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在奇次Legendre多项式零点上的$(0,2)^*$插值

谢四清

(中国科学院武汉数学物理研究所， 武汉 430071)

摘 要

设P_n表示n次Legendre多项式，本文考虑多项式$(1-x^2)P_n(x)/x$（n为奇数）零点上的$(0,2)^*$插值问题，得到了这种插值的正则性，显式表达式及收敛性.

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$(0,2)^*$ Interpolation on the Zeros of Legendre Polynomials with Odd Degree*

Xie Siqing
(Wuhan Inst. of Math. Sci., Academia Sinica)

Abstract In this paper we consider the problem of regelarity and explicit representation of $(0,2)^*$ interpolation on the zeros of $(1-x^2)P_n(x)/x$ (n odd), where P_n denotes Legendre polynomial of degree n, and the problem of convergence of interpolatory polynomials.

Keywords $(0,2)^*$ interpolation, regularity, explicit representation, convergence

AMS Classification: 41A05, 41A10/CCL O241.3

1. Introduction

In 1955, J. Suranyi and P. Turán [1] initiated the problem of so-called $(0,2)$ interpolation, that is, given n distinct points $X=\{x_i\}(1\le i\le n)$, with

$$-1\le x_1<x_2<\cdots<x_n\le 1 \tag{1.1}$$

and arbitrary numbers $\{\alpha_i\}$, $\{\beta_i\}$, one wish to decide whether or not there is a polynomial Q_{2n-1} of degree $\le 2n-1$ such that

$$Q_{2n-1}(x_i)=\alpha_i,\quad Q''_{2n-1}(x_i)=\beta_i;\quad i=1,2,\cdots,n. \tag{1.2}$$

If for any $\{\alpha_i\}$ and $\{\beta_i\}$, (1.2) has an unique solution, then we call (0,2) interpolation on X as regular (otherwise singular). When X is the set of zeros of $(1-x^2)P'_{n-1}(x)$, where P_{n-1} denotes $(n-1)$th Legendre polynomial, J. Balazs and P. Turán [2] studied the problem of existence, uniqueness, explicit representation and convergence of $(0,2)$ interpolation on X. Later R. B. Saxena [3], A. Varma and his associates [4], [5] analogously consider $(0,2)^*$ interpolation and other sorts of interpolation. It should be noted that all interpolation in these papers [2]-[5] are regular if n is even, but singular if n is odd. Let $X=\{x_i\}$ $(1\le i\le n+1)$, satisfying (1.1), be the set of zeros of $(1-x^2)P_n(x)/x$ (n odd), in this paper we shall show this sort of interpolation (which is called $(0,2)^*$ interpolation), satisfying

$$\begin{aligned} &Q(x_i)=\alpha_i\ ;\quad i=1,2,\cdots,n+1\\ &Q''(x_i)=\beta_i\ ;\quad i=2,3,\cdots,n \end{aligned} \tag{1.3}$$

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is regular. In addition, we obtain the explicit form and convergence of interpolatory polynomials.

2. Preliminaries

We shall later make use of the following well known results; see [4], [6], [7] and [8]. For $-1 \leq x \leq 1$, we have

$$(1-x^2)^{\frac{1}{4}}|P_n(x)| \leq \sqrt{\frac{2}{\pi}}n^{-\frac{1}{2}}, \tag{2.1}$$

$$(1-x^2)^{\frac{3}{4}}|P_n'(x)| \leq \sqrt{2}n^{\frac{1}{2}}, \tag{2.2}$$

$$|P_n(x)| \leq 1, \tag{2.3}$$

$$(1-x^2)^{\frac{1}{2}}|P_n'(x)| \leq n, \tag{2.4}$$

$$|P_n(x)/x| \leq 2n^{\frac{1}{2}} \quad (n\ odd), \tag{2.5}$$

$$|P_n'(0)| > (2\sqrt{2})^{-1}(1-x_i^2)^{\frac{3}{4}}|P_n'(x_i)| \quad (n\ odd), \tag{2.6}$$

$$|P_n'(x_i)| \sim (i-1)^{-\frac{3}{2}}n^2, \quad i = 2, \cdots, \frac{1}{2}(n+1), \tag{2.7}$$

$$|P_n'(x_i)| \sim (n-i+2)^{-\frac{3}{2}}n^2, \quad i = \frac{1}{2}(n+3), \cdots, n, \tag{2.8}$$

$$(1-x_i^2) > \frac{(i-1)^2}{n^2}; \quad i = 2, \cdots, \frac{1}{2}(n+1), \tag{2.9}$$

$$(1-x_i^2) > \frac{(n-i+2)^2}{n^2}; \quad i = \frac{1}{2}(n+3), \cdots, n, \tag{2.10}$$

$$\frac{2}{n+1} < \int_{-1}^{1} \frac{P_n(t)}{\sqrt{1-t^2}}\,dt < \frac{2}{n} \quad (n\ even), \tag{2.11}$$

$$\left|\int_{-1}^{x} \frac{P_n'(t)}{\sqrt{1-t^2}}\,dt\right| \leq 21n, \tag{2.12}$$

$$\left|\int_{-1}^{x} \frac{P_n(t)}{\sqrt{1-t^2}}\,dt\right| \leq \frac{7}{n}, \tag{2.13}$$

$$(1-2t\cos\theta+t^2)^{-\frac{1}{2}} = \sum_{m=0}^{\infty} t^m P_m(\cos\theta), \quad \theta \in [0,\pi]. \tag{2.14}$$

Let

$$\alpha_r = (-1)^r \binom{-\frac{1}{2}}{r}, \tag{2.15}$$

we have

$$\alpha_r = \sqrt{\frac{2}{\pi}}\sqrt{\frac{1}{2r+\theta}}, \quad 0 < \theta < 1, \tag{2.16}$$

$$P_{2r}(0) = (-1)^r \alpha_r \ , \tag{2.17}$$

$$\int_{-1}^{1} \frac{P_{2r}(t) - P_{2r+2}(t)}{\sqrt{1-t^2}} dt = (\alpha_r^2 - \alpha_{r+1}^2)\pi \ . \tag{2.18}$$

Let

$$L_i(x) = \frac{P_n(x)}{(x-x_i)P_n'(x_i)}, \ (2 \le i \le n) \quad L_0(x) = \frac{P_n(x)}{xP_n'(0)}, \ (\text{n } odd), \quad x_0 = 0, \tag{2.19}$$

then, for $i = 0, 2, \cdots, n$, we have

$$\left|\int_{-1}^{1} \frac{L_i(t)}{\sqrt{1-t^2}} dt\right| \le \frac{12}{(1-x_i^2)^{\frac{7}{4}}[P_n'(x_i)]^2}, \tag{2.20}$$

$$\left|\int_{-1}^{1} \frac{tL_i'(t)}{\sqrt{1-t^2}} dt\right| \le \frac{12n^{\frac{3}{2}}}{(1-x_i^2)^{\frac{7}{4}}[P_n'(x_i)]^2}, \tag{2.21}$$

$$\left|(1-x^2)^{\frac{1}{4}} P_n(x) \int_{-1}^{x} \frac{L_i(t)}{\sqrt{1-t^2}} dt\right| \le \frac{|L_i(x)|}{n^{\frac{3}{2}}(1-x_i^2)^{\frac{1}{2}}} + \frac{6}{n(1-x_i^2)|P_n'(x_i)|}, \tag{2.22}$$

$$(1-x^2)^{\frac{1}{4}}|L_i(x)| \le \frac{3n}{(1-x_i^2)^{\frac{5}{4}}[P_n'(x_i)]^2}, \tag{2.23}$$

$$\left|\int_{-1}^{x} \frac{tL_i'(t)}{\sqrt{1-t^2}} dt\right| \le \frac{162n^{\frac{3}{2}}}{(1-x_i^2)^{\frac{7}{4}}[P_n'(x_i)]^2}. \tag{2.24}$$

3. Regularity

To obtain the regularity of $(0,2)^*$ interpolation satisfying (1.3) we prove the following lemmas.

Lemma 1. *For n odd we have*

$$\frac{8}{5\sqrt{n+1}} \le (-1)^{\frac{1}{2}(n-1)} \int_{-1}^{1} \frac{P_n(t)}{t\sqrt{1-t^2}} dt \le \frac{9}{2\sqrt{n+1}} \ . \tag{3.1}$$

Proof. From Christoffel formula [7, P.179] we can derive that

$$-(n+1)P_{n+1}(0) \int_{-1}^{1} \frac{P_n(t)}{t\sqrt{1-t^2}} = \sum_{r=0}^{\frac{1}{2}(n-1)} (4r+1)P_{2r}(0) \int_{-1}^{1} \frac{P_{2r}(t)}{\sqrt{1-t^2}} \triangleq I. \tag{3.2}$$

If $n = 4m-1$, then we can show that

$$I = \sum_{r=0}^{\frac{1}{4}(n-3)} \alpha_{2r} \left[(8r+1) \int_{-1}^{1} \frac{P_{4r} - P_{4r+2}}{\sqrt{1-t^2}} - \frac{8r+3}{4r+2} \int_{-1}^{1} \frac{P_{4r+2}}{\sqrt{1-t^2}}\right] \tag{3.3}$$

Using (2.15), (2.16) and (2.18) we have

$$\int_{-1}^{1} \frac{P_{4r} - P_{4r+2}}{\sqrt{1-t^2}} \ge \alpha_{2r}^2 \left(1 - \frac{(4r+1)^2}{4(2r+1)^2}\right) \pi \ge \frac{8r+3}{2(4r+1)(2r+1)^2}. \tag{3.4}$$

From (2.11), (2.16), (3.3) and (3.4) we obtain

$$I \geq \sum_{r=1}^{\frac{1}{4}(n-3)} \alpha_{2r} \frac{4r(8r+3)}{2(4r+1)(2r+1)^2} + \frac{3}{8}\pi \geq \frac{8}{5}\sqrt{\frac{2}{\pi}}. \tag{3.5}$$

On the other hand, from (2.11), (2.16) and (3.3) we get

$$I \leq \frac{3}{8}\pi + \sum_{r=1}^{\frac{1}{4}(n-3)} \alpha_{2r} \left[(8r+3)\frac{2}{4r} - (8r+5)\frac{4r+1}{4r+2}\frac{2}{4r+3}\right] \leq \frac{9}{2}\sqrt{\frac{2}{\pi}}. \tag{3.6}$$

Hence, (2.17), (3.2), (3.5) and (3.6) imply (3.1). If $n = 4m+1$, we can analogously obtain the required result.

Lemma 2. *For n odd we have*

$$2n \leq \int_{-1}^{1} \frac{P_n'(t)}{\sqrt{1-t^2}}\, dt \leq 3n. \tag{3.7}$$

Proof . From (2.14) we have

$$t(1-2t\cos\theta + t^2)^{-\frac{3}{2}} = \sum_{m=0}^{\infty} P_m'(\cos\theta)t^m, \tag{3.8}$$

moreover, we know

$$\begin{aligned}(1-2t\cos\theta+t^2)^{-\frac{3}{2}} &= (1-te^{i\theta})^{-\frac{3}{2}}(1-te^{-i\theta})^{-\frac{3}{2}} \\ &= \sum_{m=0}^{\infty}\left[\sum_{k=0}^{m}\binom{-\frac{3}{2}}{k}\binom{-\frac{3}{2}}{m-k}(-1)^m t^m e^{i(2k-m)\theta}\right].\end{aligned} \tag{3.9}$$

Using (3.8) and (3.9) we obtain

$$\int_{-1}^{1}\frac{P_n'}{\sqrt{1-t^2}} = \int_0^{\pi} P_n'(\cos\theta)\, d\theta = \pi\binom{-\frac{3}{2}}{\frac{n-1}{2}}^2 = \pi n^2 \alpha_{\frac{n-1}{2}}^2. \tag{3.10}$$

From (3.10) and (2.16) we get (3.7).

Using lemma 1 and 2 we derive

Lemma 3. *If n odd, then*

$$\frac{1}{5}n^{\frac{3}{2}} \leq (-1)^{\frac{1}{2}(n+1)}\left[\int_{-1}^{1}\frac{P_n'(t)}{\sqrt{1-t^2}}\, dt - (n^2+n-1)\int_{-1}^{1}\frac{P_n(t)}{t\sqrt{1-t^2}}\, dt\right] \leq 5n^{\frac{3}{2}}. \tag{3.11}$$

Theorem 1. *Let n be odd and $-1 = x_1 < x_2 < \cdots < x_n < x_{n+1} = 1$ be the set of zeros of $w_n(x)/x = (1-x^2)P_n(x)/x$, then $(0,2)^*$ interpolation on $X = \{x_i\}$ is regular.*

Proof . It is sufficient if we prove that when the system (1.3) is homogenous, i.e.

$$\begin{aligned} Q_{2n-1}(x_i) &= 0; \qquad i = 1, 2, \cdots, n+1, \\ Q''_{2n-1}(x_i) &= 0; \qquad i = 2, 3, \cdots, n, \end{aligned} \tag{3.12}$$

the only solution is $Q_{2n-1}(x) = 0$. Clearly, from (3.12) we can write $Q_{2n-1}(x) = w_n(x)q(x)/x$, where q is a polynomial of degree $\leq n-2$ in x.

Using the fact that $Q''_{2n-1}(x_j) = 0,\ j = 2, \cdots, n$, leads to

$$x_j(1 - x_j^2)q'(x_j) - q(x_j) = 0; \quad j = 2, \cdots, n. \tag{3.13}$$

So we have, with constants a_0 and a_1,

$$x(1 - x^2)q'(x) - q(x) = \left(\frac{a_0}{x} + a_1\right) P_n(x). \tag{3.14}$$

Differentiating (3.14) once, then putting $x = 0$, we derive $a_1 = 0$. Solving the differential equation (3.14) we get

$$q = a_0 \left[-\frac{1}{2}\left(\frac{P_n}{x} - P'_n\right) + \frac{1}{2}\frac{x}{\sqrt{1-x^2}}\left(\int_{-1}^{x} \frac{P'_n}{\sqrt{1-t^2}} - (n^2 + n - 1)\int_{-1}^{x} \frac{P_n}{t\sqrt{1-t^2}}\right)\right]. \tag{3.15}$$

Putting $x \to 1^-$ in (3.15), from lemma 3 we know $a_0 = 0$, so $q(x) = 0$, hence $Q_{2n-1}(x) = 0$.

4. Explicit representation

Clearly, we can write $Q_{2n-1}(x)$ satisfying (1.3) as

$$Q_{2n-1}(x) = \sum_{i=1}^{n+1} \alpha_i r_i(x) + \sum_{i=2}^{n} \beta_i p_i(x), \tag{4.1}$$

where r_i $(1 \leq i \leq n+1)$ and p_i $(2 \leq i \leq n)$ are fundamental polynomials of degree $\leq 2n-1$, satisfying

$$\begin{aligned} r_i(x_j) &= \delta_{ij}, \qquad p_i(x_j) = 0; \qquad j = 1, 2, \cdots, n+1, \\ r''_i(x_j) &= 0, \qquad p''_i(x_j) = \delta_{ij}; \qquad j = 2, 3, \cdots, n. \end{aligned} \tag{4.2}$$

Let

$$l_i(x) = \frac{x_i P_n(x)}{x(x - x_i)P'_n(x_i)}, \quad i = 2, \cdots, n.$$

We have

Theorem 2. *For $2 \leq i \leq n$ we have*

$$\begin{aligned} r_i = {} & \frac{(1-x^2)l_i}{(1-x_i^2)P'_n(x_i)}\left(P'_n - \frac{1}{2}P_n\right) - \frac{A_i(1-x^2)P_n}{4x(1-x_i^2)P'_n(x_i)}\left(\frac{P_n}{x} + P'_n\right) \\ & + \frac{A_i(1-x^2)^{\frac{1}{2}}P_n}{4(1-x_i^2)P'_n(x_i)}\left[\int_{-1}^{x} \frac{P'_n}{\sqrt{1-t^2}} - (n^2+n-1)\int_{-1}^{x} \frac{P_n}{t\sqrt{1-t^2}}\right] \\ & + \frac{n(n+1)(1-x^2)^{\frac{1}{2}}P_n}{2(1-x_i^2)P'_n(x_i)}\int_{-1}^{x} \frac{l_i}{\sqrt{1-t^2}} - \frac{(1-x^2)^{\frac{1}{2}}P_n}{2(1-x_i^2)P'_n(x_i)}\int_{-1}^{x} \frac{t l'_i}{\sqrt{1-t^2}}, \end{aligned} \tag{4.3}$$

where

$$A_i\left[\int_{-1}^{1}\frac{P_n'(t)}{\sqrt{1-t^2}}\,dt-(n^2+n-1)\int_{-1}^{1}\frac{P_n(t)}{t\sqrt{1-t^2}}\,dt\right] = 2\int_{-1}^{1}\frac{tl_i'(t)}{\sqrt{1-t^2}}\,dt-2n(n+1)\int_{-1}^{1}\frac{l_i(t)}{\sqrt{1-t^2}}\,dt, \tag{4.4}$$

and

$$\begin{aligned} r_1 = {} & \frac{1-x}{2}\left(\frac{P_n}{x}\right)^2+\frac{(1-x^2)P_n}{2x}\left[(1+A_1)P_n'+P_n+A_1\frac{P_n}{x}\right] \\ & +\frac{1}{2}(1-x^2)^{\frac{1}{2}}P_n\int_{-1}^{x}\frac{tP_n'}{\sqrt{1-t^2}}-\frac{1}{2}(2+A_1)(1-x^2)^{\frac{1}{2}}P_n\int_{-1}^{x}\frac{P_n'}{\sqrt{1-t^2}} \\ & +\frac{1}{2}\left[(n^2+n-1)A_1+n(n+1)\right](1-x^2)^{\frac{1}{2}}P_n\int_{-1}^{x}\frac{P_n}{t\sqrt{1-t^2}}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} r_{n+1} = {} & \frac{1+x}{2}\left(\frac{P_n}{x}\right)^2+\frac{(1-x^2)P_n}{2x}\left[(1+A_{n+1})P_n'-P_n+A_{n+1}\frac{P_n}{x}\right] \\ & -\frac{1}{2}(1-x^2)^{\frac{1}{2}}P_n\int_{-1}^{x}\frac{tP_n'}{\sqrt{1-t^2}}-\frac{1}{2}(2+A_{n+1})(1-x^2)^{\frac{1}{2}}P_n\int_{-1}^{x}\frac{P_n'}{\sqrt{1-t^2}} \\ & +\frac{1}{2}\left[(n^2+n-1)A_{n+1}+n(n+1)\right](1-x^2)^{\frac{1}{2}}P_n\int_{-1}^{x}\frac{P_n}{t\sqrt{1-t^2}}, \end{aligned} \tag{4.6}$$

where $A_{n+1}=A_1$ and

$$A_1\left[\int_{-1}^{1}\frac{P_n'}{\sqrt{1-t^2}}-(n^2+n-1)\int_{-1}^{1}\frac{P_n}{t\sqrt{1-t^2}}\right] = n(n+1)\int_{-1}^{1}\frac{P_n}{t\sqrt{1-t^2}}-2\int_{-1}^{1}\frac{P_n'}{\sqrt{1-t^2}}. \tag{4.7}$$

For $2\le i\le n$, we have

$$\begin{aligned} p_i = {} & \frac{(1-x^2)P_n}{2P_n'(x_i)}\int_{-1}^{x}\frac{l_i}{\sqrt{1-t^2}}-\frac{B_i}{2}\frac{(1-x^2)P_n}{x}\left(\frac{P_n}{x}+P_n'\right) \\ & +\frac{B_i}{2}(1-x^2)^{\frac{1}{2}}P_n\left[\int_{-1}^{x}\frac{P_n'}{\sqrt{1-t^2}}-(n^2+n-1)\int_{-1}^{x}\frac{P_n}{t\sqrt{1-t^2}}\right], \end{aligned} \tag{4.8}$$

where

$$B_i\left[\int_{-1}^{1}\frac{P_n'}{\sqrt{1-t^2}}-(n^2+n-1)\int_{-1}^{1}\frac{P_n}{t\sqrt{1-t^2}}\right]=-\frac{1}{P_n'(x_i)}\int_{-1}^{1}\frac{l_i}{\sqrt{1-t^2}}. \tag{4.9}$$

Proof. We omit the details of proof, or we can check directly that the repressions in this theorem are polynomials of required degree satisfying (4.2).

5. Estimation of fundamental polynomials r_i and p_i and convergence of interpolatory polynomials

Firstly, we note the following relation

$$l_i = L_i - \frac{P_n'(0)}{P_n'(x_i)} L_0; \quad i = 2, \cdots, n. \tag{5.1}$$

Lemma 4. *For $2 \le i \le n$, we have*

$$\left| \int_{-1}^{1} \frac{l_i(t)}{\sqrt{1-t^2}}\, dt \right| \le \frac{12}{(1-x_i^2)^{\frac{7}{4}}[P_n'(x_i)]^2} + \frac{12}{|P_n'(x_i)P_n'(0)|}, \tag{5.2}$$

$$\left| \int_{-1}^{1} \frac{t l_i'(t)}{\sqrt{1-t^2}}\, dt \right| \le \frac{12n^{\frac{3}{2}}}{(1-x_i^2)^{\frac{7}{4}}[P_n'(x_i)]^2} + \frac{12n^{\frac{3}{2}}}{|P_n'(x_i)P_n'(0)|}, \tag{5.3}$$

$$|A_i| \le \frac{105 n^{\frac{1}{2}}}{(1-x_i^2)^{\frac{7}{4}}[P_n'(x_i)]^2}, \tag{5.4}$$

$$|B_i| \le \frac{210}{n^{\frac{3}{2}}(1-x_i^2)^{\frac{7}{4}}[P_n'(x_i)]^2}. \tag{5.5}$$

Proof. (5.2) and (5.3) follow from relation (5.1), inequalities (2.20) and (2.21). (5.4) follows from (5.2), (5.3), (4.4), (3.12), (3.13) and (2.6). (5.5) can be obtained analogously.

Lemma 5. *For $-1 \le x \le 1$ and $2 \le i \le n$, we have*

$$\left| (1-x^2)^{\frac{1}{2}} P_n \int_{-1}^{x} \frac{l_i}{\sqrt{1-t^2}} \right| \le \frac{30}{n^{\frac{1}{2}}(1-x_i^2)^{\frac{7}{4}}[P_n'(x_i)]^2}, \tag{5.6}$$

$$\left| \int_{-1}^{x} \frac{t l_i'}{\sqrt{1-t^2}} \right| \le \frac{560 n^{\frac{3}{2}}}{(1-x_i^2)^{\frac{7}{4}}[P_n'(x_i)]^2}, \tag{5.7}$$

$$\left| (1-x^2)^{\frac{1}{2}} P_n \int_{-1}^{x} \frac{P_n}{t\sqrt{1-t^2}} \right| \le \frac{12}{n}. \tag{5.8}$$

Proof. From (5.1), (2.22) and (2.23) we have

$$\begin{aligned} \left| (1-x^2)^{\frac{1}{2}} P_n \int_{-1}^{x} \frac{l_i}{\sqrt{1-t^2}} \right| &\le \frac{3}{n^{\frac{1}{2}}(1-x_i^2)^{\frac{7}{4}}[P_n'(x_i)]^2} \\ &+ \frac{3}{n^{\frac{1}{2}}|P_n'(0)P_n'(x_i)|} + \frac{12}{n(1-x_i^2)|P_n'(x_i)|}. \end{aligned} \tag{5.9}$$

With the help of (2.2), (2.6) and (5.9), we obtain (5.6). From (5.1), (2.24) and (2.6) we get (5.7). Moreover, using (2.22), (2.23) and the inequality $|P_n'(0)| > \frac{1}{2} n^{\frac{1}{2}}$, we obtain (5.8).

Lemma 6. *For $-1 \le x \le 1$ and $2 \le i \le n$ we have*

$$|r_i| \le 867 n^{\frac{3}{2}} (1 - x_i^2)^{-\frac{11}{4}} |P_n'(x_i)|^{-3}, \tag{5.10}$$

and

$$\sum_{i=2}^{n} |r_i(x)| \le 1156 n \log n, \tag{5.11}$$

$$\sum_{i=2}^{n} (1 - x_i^2)^{\frac{1}{2}} |r_i(x)| \le 867 n. \tag{5.12}$$

Moreover

$$|r_1| \le 720 n, \tag{5.13}$$

$$|r_{n+1}| \le 720 n. \tag{5.14}$$

Proof. For sake of brevity we write (4.3) as

$$r_i = I_1 + I_2 + I_3 + I_4 + I_5. \tag{5.15}$$

From (2.1), (2.2), (2.6) and (2.23) we have

$$|I_1| \le 23 n^{\frac{3}{2}} (1 - x_i^2)^{-\frac{9}{4}} |P_n'(x_i)|^{-3}. \tag{5.16}$$

Using (2.2), (2.5) and (5.4) we obtain

$$|I_2| \le 184 n^{\frac{3}{2}} (1 - x_i^2)^{-\frac{11}{4}} |P_n'(x_i)|^{-3}. \tag{5.17}$$

With the help of (2.1), (2.12), (5.4) and (5.8) we derive

$$|I_3| \le 525 n^{\frac{3}{2}} (1 - x_i^2)^{-\frac{11}{4}} |P_n'(x_i)|^{-3}. \tag{5.18}$$

Analogously, from (5.6), (5.7) and (2.1) we have

$$|I_4| \le 15 n^{\frac{3}{2}} (1 - x_i^2)^{-\frac{11}{4}} |P_n'(x_i)|^{-3}, \tag{5.19}$$

$$|I_5| \le 224 n (1 - x_i^2)^{-\frac{11}{4}} |P_n'(x_i)|^{-3}. \tag{5.20}$$

From (5.15)–(5.20) we obtian (5.10). From (5.10) and (2.7)–(2.10) we obtian (5.11) as well as (5.12). Similarly we can prove (5.13) and (5.14).

Lemma 7. *For $-1 \le x \le 1$ and $2 \le i \le n$, we have*

$$|p_i| \le 2700 n^{-\frac{1}{2}} (1 - x_i^2)^{-\frac{7}{4}} |P_n'(x_i)|^{-3}, \tag{5.21}$$

and

$$\sum_{i=2}^{n} (1 - x_i^2)^{-\frac{1}{2}} |p_i| \le 2700 n^{-1}. \tag{5.22}$$

Proof. (5.21) can be easily obtianed by the help of (5.5), (5.6), (5.8), (2.1), (2.2), (2.5), and (2.12). (5.22) follows from (5.21) and (2.7)–(2.10).

Theorem 3. *Let $f(x)$ be a continuous function satisfying the Zygmund condition*

$$|f(x+h)-2f(x)+f(x-h)|=o(h)$$

in $[-1,1]$ and let $|\beta_i|\le o(n)(1-x_i^2)^{-\frac{1}{2}}$, then the sequence of interpolatory polynomials $Q_{2n-1}(x)$ in (4.1) (with $\alpha_i=f(x_i)$) converges uniformly to $f(x)$ in $[-1,1]$.

Proof. The proof of this theorem could be obtianed on the same lines as in [4] and is omitted here.

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在奇次Legendre多项式零点上的$(0,2)^*$插值

谢四清

(中国科学院武汉数学物理研究所，　武汉　430071)

摘　要

设P_n表示n次Legendre多项式，本文考虑多项式$(1-x^2)P_n(x)/x$（n为奇数）零点上的$(0,2)^*$插值问题，得到了这种插值的正则性，显式表达式及收敛性.