关于 Matsuoka 算子逼近度的渐近表示*

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摘　要　本文得出用 Matsuoka 算子 $J_{n,p,q}(f;x)$ 逼近函数 $(\in B_{2\pi})$ 的逼近度关于准左、右导数及二阶准左、右导数的渐近表示式.

关键词　Matsuoka 算子，逼近度，准左、右导数[1].
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设 $f(x)$ 是周期为 2π 的可积函数，$n,p,q\in N$(自然数集)，称

$$J_{n,p,q}(f;x)=\int_{-\pi}^{\pi}f(t)\frac{\left(\sin n\dfrac{t-x}{2}\right)^{2p}}{\left(\sin\dfrac{t-x}{2}\right)^{2q}}dt\Big/\int_{-\pi}^{\pi}\frac{\left(\sin\dfrac{nt}{2}\right)^{2p}}{\left(\sin\dfrac{t}{2}\right)^{2q}}dt \tag{1}$$

为 Matsuoka 算子. 当 $p=q=1$ 时，它为 Fejer 算子；当 $p=q=2$ 时，它为 Jackson 算子；对它们一些专著研究得很深刻；但对一般的 p,q 涉及较少. 吴顺唐[3]及 Natanson[4] 曾分别得出用 Jackson 算子逼近函数 $f(x)$ 的逼近度关于准左、右导数及二阶准左、右导数的渐近表示式. 本文得出用 Matsuoka 算子逼近函数 $f(x)$ $(\in B_{2\pi})$ 的逼近度关于准左、右导数及二准左、右导数的渐近表示式.

引理1　设 $n,p,q\in N$，$p\geqslant q$，有等式

$$\frac{\sin^{2p}nt}{\sin^{2q}t}=B_{0,p,q}^{(n)}+2\sum_{k=1}^{np-q}B_{k,p,q}^{(n)}\cos 2kt \tag{2}$$

其中：

$$B_{kpq}=\left(\frac{-1}{4}\right)^{p-q}\cdot\begin{cases}\left[\sum\limits_{j=0}^{p-1}(-1)^j\dbinom{2p}{j}\dbinom{(p-j)n+q-1-k}{2q-1}\right], & 0\leqslant k\leqslant n-q;\\ \left[\sum\limits_{j=0}^{p-2}(-1)^j\dbinom{2p}{j}\dbinom{(p-j)n+q-1-k}{2q-1}\right], & \\ \qquad n-q+1\leqslant k\leqslant 2n-q; & \\ \cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots & \\ \left[\sum\limits_{j=0}^{1}(-1)^j\dbinom{2p}{j}\dbinom{(p-j)n+q-1-k}{2q-1}\right], & \\ \qquad (p-2)n-q+1\leqslant k\leqslant(p-1)n-q; & \\ \dbinom{pn+q-1-k}{2q-1}, & (p-1)n-q+1\leqslant k\leqslant pn-q.\end{cases} \tag{2a}$$

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证明 选取生成函数：$(1-x)^{-2q}=\sum\limits_{k'=0}^{\infty}\binom{k'+2q-1}{2q-1}x^{k'}$，

$$(1-x^n)^{2p}(1-x)^{-2q}=\left[\sum_{j=0}^{2p}(-1)^j\binom{2p}{j}x^{jn}\right]\left[\sum_{k'=0}^{\infty}\binom{k'+2q-1}{2q-1}x^{k'}\right]$$

$$=\sum_{k'=0}^{n-1}\binom{k'+2q-1}{2q-1}x^{k'}+\sum_{k'=n}^{2n-1}\left[\binom{k'+2q-1}{2q-1}-\binom{2p}{1}\binom{k'+2q-1-n}{2q-1}\right]x^{k'}$$

$$+\cdots+\sum_{k'=(p-2)n}^{(p-1)n-1}\left[\sum_{j=0}^{p-2}(-1)^j\binom{2p}{j}\binom{k'+2q-1-jn}{2q-1}\right]x^{k'}$$

$$+\sum_{k'=(p-1)n}^{pn-1}\left[\sum_{j=0}^{p-1}(-1)^j\binom{2p}{j}\binom{k'+2q-1-jn}{2q-1}\right]x^{k'}+\cdots. \tag{3}$$

因$(1-x^n)^{2p}(1-x)^{-2q}$是$2(pn-q)$次对称多项式，其系数关于中心项x^{np-q}是对称的；故由(3)式可得：

$$\frac{(1-x^n)^{2p}}{(1-x)^{2q}}=\left\{\sum_{k'=0}^{n-1}\binom{k'+2q-1}{2q-1}+\sum_{k'=n}^{2n-1}\left[\binom{k'+2q-1}{2q-1}-\binom{2p}{1}\binom{k'+2q-1}{2q-1}\right]\right.$$

$$+\cdots+\sum_{k'=(p-2)n}^{(p-1)n-1}\left[\sum_{j=0}^{p-2}(-1)^j\binom{2p}{j}\binom{k'+2q-1-jn}{2q-1}\right]$$

$$\left.+\sum_{k'=(p-1)n}^{pn-q-1}\left[\sum_{j=0}^{p-1}(-1)^j\binom{2p}{j}\binom{k'+2q-1-jn}{2q-1}\right]\right\}\left[x^{k'}+x^{2pn-2q-k'}\right]$$

$$+\left[\sum_{j=0}^{p-1}(-1)^j\binom{2p}{j}\binom{(p-j)n+q-1}{2q-1}\right]x^{pn-q} \tag{4}$$

$$\overset{k'=pn-q-k}{=}\left\{\sum_{k=(p-1)n-q+1}^{pn-q}\binom{pn+q-1-k}{2q-1}\right.$$

$$+\sum_{k=(p-2)n-q+1}^{(p-1)n-q}\left[\sum_{j=0}^{1}(-1)^j\binom{2p}{j}\binom{(p-j)n+q-1-k}{2q-1}\right]$$

$$+\cdots+\sum_{k=n-q+1}^{2n-q}\left[\sum_{j=0}^{p-2}(-1)^j\binom{2p}{j}\binom{(p-j)n+q-1-k}{2q-1}\right]$$

$$\left.+\sum_{k=1}^{n-q}\left[\sum_{j=0}^{p-1}(-1)^j\binom{2p}{j}\binom{(p-j)n+q-1-k}{2q-1}\right]\right\}\left[x^{np-q-k}+x^{np-q+k}\right]$$

$$+\left[\sum_{j=0}^{p-1}(-1)^j\binom{2p}{j}\binom{(p-j)n+q-1}{2q-1}\right]x^{pn-q}. \tag{5}$$

又

$$\frac{\sin^{2p}nt}{\sin^{2q}t}=\frac{\left[(e^{int}-e^{-int})/2i\right]^{2p}}{\left[(e^{it}-e^{-it})/2i\right]^{2q}}=\left(\frac{-1}{4}\right)^{p-q}\cdot e^{-(pn-q)2ti}\frac{(1-e^{2nti})^{2p}}{(1-e^{2ti})^{2q}}. \tag{6}$$

令$x=e^{2ti}$，则由(5)，(6)式及 Euler 公式可得(2)，(2a)式．

系 1 $$\int_{-\pi}^{\pi}\frac{\sin^{2p}nt}{\sin^{2q}t}dt=2\pi B_{0pq}^{(n)}=2\pi\left(\frac{-1}{4}\right)^{p-q}\left[\sum_{j=0}^{p-1}(-1)^j\binom{2p}{j}\binom{(p-j)n+q-1}{2q-1}\right] \tag{7}$$

系 2 $$a_{pq}=\lim_{n\to\infty}B_{0pq}^{(n)}/n^{2q-1}=\left(\frac{-1}{4}\right)^{p-q}\left[\sum_{j=0}^{p-1}(-1)^j\binom{2p}{j}(p-j)^{2q-1}/(2q-1)!\right] \tag{8}$$

引理 2

i) $$\int_0^n \frac{\sin^{2p}t}{t}dt = C_{2p}^p \ln n/2^{2p} + O(1),\ (n \to \infty). \tag{9}$$

ii) $$A_{1pq} = \int_0^\infty \frac{\sin^{2p}t}{t^{2q-1}}dt = \frac{\sum_{k=2}^{p}(-1)^{k+q}k^{2q-2}\ln k \cdot C_{2p}^{p-k}}{(2q-2)!\,2^{2p-2q+1}},\ (p \geqslant q \geqslant 2), \tag{10}$$

$$B_{1pq} = \int_0^\infty \frac{\sin^{2p}t}{t^{2q-2}}dt = \frac{\sum_{k=1}^{p}(-1)^{k+q-1}k^{2q-3}\ln k \cdot C_{2p}^{p-k}\pi}{(2q-3)!\,2^{2p-2q+3}},\ (p \geqslant q \geqslant 2). \tag{11}$$

证明 i)由分部积分易得：

$$\int_0^{-\infty}\cos 2kte^{-tx}dt = \frac{x}{x^2+4k^2}, \tag{12}$$

$$\lim_{n\to\infty}\int_0^n \frac{\sin^{2p}t}{t}dt = \lim_{n\to\infty}\int_{\frac{1}{n}}^n \sin^{2p}t\left(\int_0^\infty e^{-tx}dx\right)dt = \lim_{n\to\infty}\int_{\frac{1}{n}}^n\left(\int_0^\infty \sin^{2p}te^{-tx}dt\right)dx$$

$$= \lim_{n\to\infty}\frac{1}{2^{2p}}\int_0^n\left[\int_0^\infty\left(2\sum_{k=1}^{p}(-1)^kC_{2p}^{p-k}\cos 2kt + C_{2p}^p\right)e^{-tx}dt\right]dx$$

$$= \lim_{n\to\infty}\frac{1}{2^{2p}}\int_0^n\left[2\sum_{k=1}^{p}\frac{(-1)^kC_{2p}^{p-k}x}{x^2+4k^2} + \frac{C_{2p}^p}{x}\right]dx$$

$$= \frac{1}{2^{2p}}\lim_{n\to\infty}\left[\sum_{k=1}^{p}(-1)^kC_{2p}^{p-k}\ln(x^2+4k^2) + C_{2p}^p\ln x\right]\Bigg|_{\frac{1}{n}}^{n}.$$

因 $2\sum_{k=1}^{p}(-1)^kC_{2p}^{p-k} + C_{2p}^p = 0$，故有(9)式成立：

$$\int_0^n \frac{\sin^{2p}t}{t}dt = C_{2p}^p\ln n/2^{2p} + O(1),\ (n\to\infty). \tag{9}$$

(ii) 当 $p\geqslant q\geqslant 2$ 时，由分部积分，有

$$\int_0^\infty \frac{\sin^{2p}t}{t^{2q-1}}dt = -\left[\frac{\sin^{2p}t}{(2q-2)t^{2q-2}} + \frac{(\sin^{2p}t)'}{(2q-2)(2q-3)t^{2q-3}} + \cdots + \frac{(\sin^{2p}t)^{(2q-3)}}{(2q-2)!\,t}\right]\Bigg|_0^\infty$$
$$+ \int_0^\infty \frac{(\sin^{2p}t)^{(2q-2)}}{(2q-2)!\,t}dt. \tag{13}$$

因

$$\frac{(\sin^{2p}t)^{(m)}}{t^{2q-2-m}}\Bigg|_0^\infty = \frac{\sin^{2p-m}t \cdot Q_m(\sin t,\cos t)}{t^{2q-2-m}}\Bigg|_0^\infty = 0,\ (m = 0,1,\cdots,2q-3). \tag{14}$$

其中 $Q_m(\sin t,\cos t)$表示一 $\sin t,\cos t$ 的多项式. 这样由(13),(14)有

$$\int_0^\infty \frac{\sin^{2p}t}{t^{2q-1}}dt = \int_0^\infty \frac{\left(2\sum_{k=1}^{p}(-1)^kC_{2p}^{p-k}\cos 2kt + C_{2p}^p\right)^{(2q-2)}}{(2q-2)!\,2^{2p}t}dt. \tag{15}$$

由 $(\sin^{2p}t)^{(2m)} = \left(2\sum_{k=1}^{p}(-1)^kC_{2p}^{p-k}\cos 2kt + C_{2p}^p\right)^{(2m)}/2^{2p}$，令 $t=0$，则得

$$\sum_{k=1}^{p}(-1)^kC_{2p}^{p-k}k^{2m} = 0\ (1\leqslant m\leqslant p-1). \tag{16}$$

我们有公式

$$\int_0^\infty \frac{\cos at - \cos bt}{t}dt = \ln \frac{b}{a}. \tag{17}$$

由(15),(16),(17)式,有

$$A_{1pq} = \int_0^\infty \frac{\sin^{2p}t}{t^{2q-1}}dt = \int_0^\infty \frac{2\sum_{k=1}^{p}(-1)^{k+q-1}C_{2p}^{p-k}(2k)^{2q-2}\cos 2kt}{(2q-2)!\,2^{2p}t}dt$$

$$= \frac{\sum_{k=1}^{p}(-1)^{k+q}C_{2p}^{p-k}k^{2q-2}\ln 2k}{(2q-2)!\,2^{2p-2q+1}} = \frac{\sum_{k=2}^{p}(-1)^{k+q}C_{2p}^{p-k}k^{2q-2}\ln k}{(2q-2)!\,2^{2p-2q+1}} \tag{10}$$

同(13),(15)式一样,及 $\int_0^\infty \frac{\sin t}{t} = \frac{\pi}{2}$,有

$$\int_0^\infty \frac{\sin^{2p}t}{t^{2q-2}}dt = \int_0^\infty \frac{\left(2\sum_{k=1}^{p}(-1)^k C_{2p}^{p-k}\cos 2kt + C_{2p}^{p}\right)^{(2q-3)}}{(2q-3)!\,2^{2p}t}$$

$$= \int_0^\infty \frac{2\sum_{k=1}^{p}(-1)^{k+q-1}C_{2p}^{p}(2k)^{2q-3}\sin 2kt}{(2q-3)!\,2^{2p}t}dt \tag{18}$$

$$B_{1pq} = \int_0^\infty \frac{\sin^{2p}t}{t^{2q-2}}dt = \frac{\sum_{k=1}^{p}(-1)^{k+q-1}C_{2p}^{p-k}k^{2q-3}\pi}{(2q-3)!\,2^{2p-2q+3}}. \tag{11}$$

引理 3[1](渐近关系转化定理) 设 $\varphi(t)$是非负连续偶函数,且至多有 $\varphi(0)=0$,$\{L_n\}_{n\in N}$是 $B(D)$上正线性算子列,对 $x\in D$ 适合如下条件:当 $n\to\infty$时有

i) $L_n(1;x)-1=o_x(L_n(\varphi(t-x);x))$;

ii) 对每个 $\delta>0$,记 $\lambda_\delta(t,x)=\begin{cases}1; & |t-x|\geqslant\delta>0\\ 0; & |t-x|<\delta\end{cases}$,有 $L_n(\lambda_\delta(t,x),x)=o_x(L_n(\varphi(t-x),x))$;若 $f\in B(D)$,且当 $t\to x_\pm$ 时,有 $\lim\limits_{t\to x_\pm}\frac{f(t)-f(x_\pm)}{\varphi(t-x)}=G_\pm(x)$,其中 $f(x_-)$,$f(x_+)$分别是 $f(t)$在 x 点的左、右极限,而 $G_\pm(x)$是有限的且适合如下条件:当 $n\to\infty$时有

iii) $(f(x_+)-f(x_-))L_n(S_{gn}(t-x),x)=o_x(L_n(\varphi(t-x),x))$;

iv) $(G_+(x)-G_-(x))L_n(\varphi(t-x),S_{gn}(t-x),x)=o_x(L_n(\varphi(t-x),x))$.

则有

$$L_n(f;x)-\frac{f(x_+)+f(x_-)}{2}\sim\frac{G_+(x)+G_-(x)}{2}\ln(\varphi(t-x),x),\ (n\to\infty). \tag{19}$$

定理 1 设 $f(x)\in B_{2\pi}(D)$,且在 x 点具有准左、右导数 $f'_-(x)$和 $f'_+(x)$[1],$q\leqslant p\in N$,记 Matsuoka 算子为 $J_{n,p,q}(f;x)$,则有:

(i) $$J_{n,p,1}(f;x)-\frac{f(x_+)+f(x_-)}{2}=\frac{2C_{2p}^{p}\ln n}{2^{2p}a_{p1}n\pi}(f'_+(x)-f'_-(x))+O_x(n^{-1}),\ (n\to\infty); \tag{20}$$

(ii) $$J_{n,p,2}(f;x)-\frac{f(x_+)+f(x_-)}{2}=\frac{2A_{1p2}}{a_{p2}n\pi}(f'_+(x)-f'_-(x))+O_x(n^{-3}\ln n),\ (p\geqslant2,n\to\infty); \tag{21}$$

(iii) $$J_{n,p,q}(f;x)-\frac{f(x_+)+f(x_-)}{2}=\frac{2A_{1pq}}{a_{pq}n\pi}(f'_+(x)-f'_-(x))+O_x(n^{-3}),\ (p\geqslant q\geqslant3,n\to\infty), \tag{22}$$

其中 A_{1pq} 和 a_{pq} 分别由(10)和(8)式表示.

证明 设 $f(x)\leqslant B_{2\pi}(D)$，且在 x 点具有准左、右导数 $f_-(x)$，$f_+(x)$. 取 $\varphi(t)=|t|$，又 $|t|\sim 2\left|\sin\frac{t}{2}\right|(t\to 0)$，由引理 1,2,3 有：

$$G_-(x)=\lim_{t\to x_-}\frac{f(t)-f(x_-)}{|t-x|}=-f_-(x);\quad G_+(x)=\lim_{t\to x_+}\frac{f(t)-f(x_+)}{|t-x|}=f_+(x);$$

$$J_{n,p,q}\left(2\left|\sin\frac{t-x}{2}\right|;x\right)=\frac{1}{2\pi B_{0pq}^{(n)}}\int_{-\pi}^{\pi}2\left|\sin\frac{t}{2}\right|\frac{\sin^{2p}\frac{nt}{2}}{\sin^{2q}\frac{t}{2}}dt=\frac{4}{B_{0pq}^{(n)}\pi}\int_0^{\frac{\pi}{2}}\frac{\sin^{2p}nu}{\sin^{2q-1}u}du. \tag{23}$$

因 $(\sin u)^{-(2q-1)}=u^{-(2q-1)}(1+\sum_{m=1}^{\infty}a_mu^{2m})$，其中 a_m 是可用多项式定理求出的一些常数. 故

$$\begin{aligned}J_{n,p,q}\left(2\left|\sin\frac{t-x}{2}\right|;x\right)&=\frac{4}{\pi B_{0pq}^{(n)}}\left[\int_0^{\frac{\pi}{2}}\left(\frac{\sin^{2p}nu}{u^{2q-1}}+\sum_{m=1}^{q-1}\frac{a_m\sin^{2p}nu}{u^{2q-1-2m}}\right)du+O(1)\right]\\&=\frac{4}{\pi B_{0pq}^{(n)}}\left[n^{2q-2}\int_0^{\frac{n\pi}{2}}\frac{\sin^{2p}t}{t^{2q-1}}dt+\sum_{l=2}^{q}a_{l-1}n^{2q-2l}\int_0^{\frac{n\pi}{2}}\frac{\sin^{2p}t}{t^{2q-2l+1}}dt+O(1)\right]\\&=\frac{4}{\pi B_{0pq}^{(n)}}\left[n^{2q-2}\int_0^{\infty}\frac{\sin^{2p}t}{t^{2q-1}}dt+\sum_{l=2}^{q-1}a_{l-1}n^{2q-2l}\int_0^{\infty}\frac{\sin^{2p}t}{t^{2q-2l+1}}dt+a_{q-1}\int_0^{n}\frac{\sin^{2p}t}{t}dt+O(1)\right.\\&\quad\left.-n^{2q-2}\int_{\frac{n\pi}{2}}^{\infty}\frac{\sin^{2p}t}{t^{2q-1}}dt-\sum_{l=2}^{q-1}a_{l-1}n^{2q-2l}\int_{\frac{n\pi}{2}}^{\infty}\frac{\sin^{2p}t}{t^{2q-2l+1}}dt+a_{q-1}\int_{\frac{n}{2}}^{\frac{n\pi}{2}}\frac{\sin^{2p}t}{t}dt\right].\end{aligned} \tag{24}$$

因为积分 $A_{lpq}=\int_0^{\infty}\frac{\sin^{2p}t}{t^{2q-2l+1}}dt$ 收敛 $(l=1,2,\cdots,q-1;q\geqslant 2)$，其值可由引理 2 之方法计算. 由(7)，(8)，(9)，(10)及(24)式可得：

$$J_{n,p,q}\left(2\left|\sin\frac{t-x}{2}\right|;x\right)=\frac{4}{a_{pq}\pi}\left[\frac{A_{1pq}}{n}+\sum_{l=2}^{q-1}\frac{a_{l-1}A_{lpq}}{n^{2l-1}}+\frac{a_{q-1}C_{2p}^{p}\ln n}{2^{2p}n^{2q-1}}+O(n^{-2q+1})\right], \tag{25}$$

$$J_{n,p,q}\left(2\left|\sin\frac{t-x}{2}\right|;x\right)=\frac{4A_{1pq}}{a_{pq}n\pi}+O(n^{-3}),\ (n\to\infty,p\geqslant q\geqslant 3), \tag{26}$$

$$J_{n,p,2}\left(2\left|\sin\frac{t-x}{2}\right|;x\right)=\frac{4A_{1p2}}{a_{p2}n\pi}+O(n^{-3}\ln n),\ (n\to\infty,p\geqslant 2). \tag{27}$$

用求(23)，(24)，(25)式之方法及(9)式可得：

$$\begin{aligned}J_{n,p,1}\left(2\left|\sin\frac{t-x}{2}\right|;x\right)&=\frac{4}{\pi B_{0p1}^{(n)}}\int_0^{\frac{\pi}{2}}\frac{\sin^{2p}nu}{\sin u}du=\frac{4}{\pi B_{0p1}^{(n)}}\left[\int_0^{\frac{\pi}{2}}\frac{\sin^{2p}nu}{u}du+O(1)\right]\\&=\frac{4}{\pi B_{0p1}^{(n)}}\left[\int_0^{n}\frac{\sin^{2p}t}{t}dt+\int_{n}^{\frac{n\pi}{2}}\frac{\sin^{2p}t}{t}dt+O(1)\right],\end{aligned}$$

$$J_{n,p,1}\left(2\left|\sin\frac{t-x}{2}\right|;x\right)=\frac{4C_{2p}^{p}\ln n}{\pi 2^{2p}a_{p1}n}+O(n^{-1})(n\to\infty). \tag{28}$$

由于 $J_{n,p,q}(1;x)=1$，$J_{n,p,q}\left(2\sin\frac{t-x}{2};x\right)=0$，$J_{n,p,q}(S_{ym}(t-x);x)=0$，所以引理 3 之条件 i)，iii)，iv)都满足，又因 $\delta_n\to 0^+(n\to\infty)$ 有

$$J_{n,p,q}(\lambda_{\delta_n}(t,x);x)=\frac{2}{\pi B_{0pq}^{(n)}}\int_{\frac{\delta_n}{2}}^{\frac{\pi}{2}}\frac{\sin^{2p}nu}{\sin^{2q}u}du<\frac{2}{\pi B_{0pq}^{(n)}}\int_{\frac{\delta_n}{2}}^{\infty}\left(\frac{\pi}{2}\right)^{2q}\frac{du}{u^{2q}}=\frac{\pi^{2q-1}}{(2q-1)B_{0pq}^{(n)}\delta_n^{2q-1}}. \tag{29}$$

对 $q\geqslant 2$，取 $\delta_n=n^{-\frac{1}{2q-1}}\to 0^+(n\to\infty)$，则由(27)，(26)和(29)式可得

$$\frac{J_{n,p,2}(\lambda_{\delta_n}(t,x);x)}{J_{n,p,2}(2|\sin\frac{t-x}{2}|;x)} \leqslant \frac{\pi^3/3B_{0,p,2}^{(n)}n^{-1}}{4A_{1p2}/a_{p2}n\pi+O(n^{-3}\ln n)} \to 0(n\to\infty, p\geqslant 2), \tag{30}$$

$$\frac{J_{n,p,2}(\lambda_{\delta_n}(t,x);x)}{J_{n,p,2}(2|\sin\frac{t-x}{2}|;x)} \leqslant \frac{\pi^{2q-1}/(2q-1)B_{0pq}^{(n)}n^{-1}}{4A_{1pq}/n\pi a_{pq}+O(n^{-3})} \to 0(n\to\infty, p\geqslant q\geqslant 3). \tag{31}$$

对 $q=1$,取 $\delta_n(\ln n)^{-\frac{1}{2}}\to 0^+$,$(n\to\infty)$,则由(28),(29)式,可得

$$\frac{J_{n,p,1}(\lambda_{\delta_n}(t,x);x)}{J_{n,p,1}(2|\sin\frac{t-x}{2}|;x)} \leqslant \frac{\pi/B_{0p1}(\ln n)^{-\frac{1}{2}}}{4C_{2p}^2\ln n/2^{2p}a_{p1}n\pi+O(n^{-1})} \to 0(n\to\infty). \tag{32}$$

由(30),(31),(32)式及文献[1]p62,引理 1.7;对每个 $\delta>0$,有

$$J_{n,p,q}(\lambda_\delta(t,x);x)=o_x(J_{n,p,q}(|t-x|;x)),(n\to\infty).$$

这样对 $J_{n,p,q}(f;x)$,引理 3 条件 ii)也满足.由引理 2,3 及(26),(27),(28)式则得:

$$J_{n,p,1}(f;x)-\frac{f(x_+)+f(x_-)}{2}=\frac{2C_{2p}^p\ln n}{\pi 2^{2p}a_{p1}n}(f_+(x)-f_-(x))+O_x(n^{-1}),\ (n\to\infty); \tag{20}$$

$$J_{n,p,2}(f;x)-\frac{f(x_+)+f(x_-)}{2}=\frac{2A_{1p2}}{\pi a_{p2}n}(f_+'(x)-f_-'(x))+O_x(n^{-3}\ln n),\ (n\to\infty, p\geqslant 2); \tag{21}$$

$$J_{n,p,q}(f;x)-\frac{f(x_+)+f(x_-)}{2}=\frac{2A_{1pq}}{\pi a_{pq}n}(f_+'(x)-f_-'(x))+O_x(n^{-3}),\ (n\to\infty, p\geqslant q\geqslant 3), \tag{22}$$

其中 A_{1pq} 和 a_{pq} 分别由(10)和(8)式计算.

定理 2 设 $f(x)\in B_{2\pi}(D)$,且在 x 点具有二阶准左、右导数 $f_-(x)$,$f_+(x)$,$2\leqslant q\leqslant p\in N$,记 Matsuoka 算子为 $J_{n,p,q}(f;x)$,则有:

$$J_{n,p,2}(f;x)-\frac{f(x_+)+f(x_-)}{2}=\frac{2B_{1p2}}{\pi a_{p2}n^2}(f_+(x)+f_-(x))+O_x(n^{-3}),(n\to\infty, p\geqslant 2); \tag{33}$$

$$J_{n,p,q}(f;x)-\frac{f(x_+)+f(x_-)}{2}=\frac{2B_{1pq}}{\pi a_{pq}n^2}(f_+(x)+f_-(x))+O_x(n^{-4}),(n\to\infty, p\geqslant q\geqslant 3), \tag{34}$$

其中 B_{1pq} 和 a_{pq} 分别由(11)和(8)式表示.

证明 设 $f(x)\in B_{2\pi}(D)$且在点 x 具有二阶准左、右导数 $f_-(x)$,$f_+(x)$,取 $\varphi(t)=\frac{1}{2}t^2$,又 $\frac{1}{2}t^2\sim 2\sin^2\frac{t}{2}(t\to 0)$,取辅助函数 $F(t)=f(t)-f^{(1)}(x)(t-x)$,其中 $f^{(1)}(x)$为 $f(x)$在点 x 的准导数,有 $G_\pm(x)=\lim\limits_{t\to x_\pm}\dfrac{f(t)-f^{(1)}(x)(t-x)-f(x_\pm)}{\frac{1}{2}(t-x)^2}=f_\pm(x)$;由引理 1,2,3,则有

$$J_{n,p,q}(2\sin^2\frac{t-x}{2};x)=\frac{1}{2\pi B_{0pq}^{(n)}}\int_{-\pi}^{\pi}2\sin^2\frac{t}{2}\frac{\sin^{2p}\frac{nt}{2}}{\sin^{2q}\frac{t}{2}}dt=\frac{4}{\pi B_{0pq}^{(n)}}\int_0^{\frac{\pi}{2}}\frac{\sin^{2p}nu}{\sin^{2q-2}u}du. \tag{35}$$

因 $(\sin u)^{-(2q-2)}=u^{-(2q-2)}(1+\sum\limits_{m=1}^{\infty}b_m u^{2m})$,其中 b_m 是可用多项式定理求出的一些系数,故

$$J_{n,p,q}(2\sin^2\frac{t-x}{2}|;x)=\frac{4}{\pi B_{0pq}^{(n)}}\left[\int_0^{\frac{\pi}{2}}(\frac{\sin^{2p}nu}{u^{2q-2}}+\sum_{l=2}^{q-1}\frac{b_{l-1}\sin^{2p}nu}{u^{2q-2l}})du+O(1)\right]$$

$$
\begin{aligned}
&= \frac{4}{\pi B_{0pq}^{(n)}}\left[n^{2q-3}\int_0^{\frac{n\pi}{2}}\frac{\sin^{2p}t}{t^{2q-2}} + \sum_{l=2}^{q-1} n^{2q-2l-1}\int_0^{\frac{n\pi}{2}}\frac{\sin^{2p}t}{t^{2q-2l}}dt + O(1)\right] \\
&= \frac{4}{\pi B_{0pq}^{(n)}}\left[n^{2q-3}\int_0^{\infty}\frac{\sin^{2p}t}{t^{2q-2}}dt + \sum_{l=2}^{q-1} b_{l-1}n^{2q-2l-1}\int_0^{\infty}\frac{\sin^{2p}t}{t^{2q-2l}}dt + O(1)\right. \\
&\quad \left. - n^{2q-3}\int_{\frac{n\pi}{2}}^{\infty}\frac{\sin^{2p}t}{t^{2q-2}}dt - \sum_{l=2}^{q-1} b_{l-1}n^{2q-2l-1}\int_{\frac{n\pi}{2}}^{\infty}\frac{\sin^{2p}t}{t^{2q-2l}}dt\right].
\end{aligned} \tag{36}
$$

因积分 $B_{lpq}=\int_0^{\infty}\frac{\sin^{2p}t}{t^{2q-2l}}dt$ 收敛($l=1,2,\cdots,q-1$),其值可用引理 2 之方法计算. 由(7),(8),(11)和(36)式,当 $n\to\infty$时有:

$$
J_{n,p,q}(2\sin^2\frac{t-x}{2};x)=\frac{4}{\pi a_{pq}}\left[\frac{B_{1pq}}{n^2}+\sum_{l=2}^{q-1}\frac{b_{l-1}B_{lpq}}{n^{2l}}+O(n^{-(2q-1)})\right]\ (n\to\infty), \tag{37}
$$

$$
=\begin{cases}\dfrac{4B_{1p2}}{\pi a_{p2}n^2}+O(n^{-3}),\ (n\to\infty,p\geqslant 2)\\[2ex] \dfrac{4B_{1pq}}{\pi a_{pq}n^2}+O(n^{-4}),\ (n\to\infty,p\geqslant q\geqslant 3).\end{cases} \tag{38}
$$

因 $J_{n,p,q}(1;x)=1$, $J_{n,p,q}(\sin(t-x);x)=J_{n,p,q}(\mathrm{sgn}(t-x);x)=J_{n,p,q}(2\sin^2\frac{t-x}{2}\mathrm{sgn}(t-x);x)=0$,对每个 $\delta_n\to 0^+$ 有 $J_{n,p,q}(\lambda_{\delta_n}(t,x);x)\leqslant \pi^{2q-1}/(2q-1)B_{0pq}^{(n)}\delta_n^{2q-1}$;选取 $\delta_n=(\ln n)^{-\frac{1}{2q-1}}\to 0^+$ $(n\to\infty$,则由(29),(38)式有

$$
\frac{J_{n,p,q}(\lambda_{\delta_n}(t,x))}{J_{n,p,q}(2\sin^2\frac{t-x}{2};x)}\leqslant\frac{\pi^{2q-1}/(2q-1)B_{0pq}^{(n)}(\ln n)^{-1}}{4B_{1pq}/\pi a_{pq}n^2+O(n^{-4})}\to 0\ (n\to\infty). \tag{39}
$$

这样对 $J_{n,p,q}(f;x)$,引理 3 的条件都满足,由引理 2,3 及(38)式,得

$$
J_{n,p,2}(f;x)-\frac{f(x_+)+f(x_-)}{2}=\frac{2B_{1p2}}{\pi a_{p2}n^2}(f''_+(x)+f''_-(x))+O_x(n^{-3}),(n\to\infty,p\geqslant 2), \tag{34}
$$

$$
J_{n,p,q}(f;x)-\frac{f(x_+)+f(x_-)}{2}=\frac{2B_{1pq}}{\pi a_{pq}n^2}(f''_+(x)+f''_-(x))+O_x(n^{-4}),(n\to\infty,p\geqslant q\geqslant 3), \tag{35}
$$

其中 B_{1pq} 和 a_{pq} 分别由(11),(8)式表示.

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The Asymptotic Representation of the Approximate Degree for the Matsuoka Operator

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Abstract

We obtain the asymptotic representation of the approximate degree for the Matsuoka operator $J_{n,p,q}(f,x)$ on $f(x)$, where $f(x)$ is assumed to have the left, right pre-derivate $f'_-(x), f'_+(x)$ and the left, right per-derivate of the second order $f''_-(x), f''_+(x)$ at x.

Keywords Matsuoka operator, approximate degree, per-derivate.