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广义有理样条的几种构造方法

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摘　要

本文利用 Thiele 倒差分方法、Padé 逼近方法、广义 Q.D. 算法及 ε- 算法等构造了几种广义有理样条函数. 此外，通过直接法构造了 $(k-1,k)$- 型广义有理样条，给出了它的行列式表示和余项表示并证明了广义有理样条算子的存在性、唯一性、齐次性及连续性.

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A Few Constructions of Generalized Rational Splines *

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Abstract The aim of this paper is to construct a few kinds of generalized interpolating rational splines by various method, including Thiele's reciprocal difference method, Padé approximant method, generalized Q.D.algorithm and generalized ε-algorithm. Among others we also establish in direct way the determinant representations and remainder representations of the generalized rational splines of type $(k-1,k)$ and prove some properties of them, such as existence, uniqueness, homogeneity and continuity of generalized rational spline operators.

Keywords interpolation, rational spline, Padé approximant, divided difference.

Classification AMS(1991) 41A/CCL O174.41

1. Introduction

Suppose $[a,b]$ is an interval which is divided into subintervals by the following partition

$$T: a=t_0<t_1<\cdots<t_{j-1}<t_j<\cdots<t_n=b.$$

Let $T_j=[t_j,t_{j+1}]$, then $[a,b]=\cup_{j=0}^{n-1}T_j$. Suppose $\delta>0$ is so small that $\delta(t_j)=[t_j-\delta,t_j+\delta]\subset T_{j-1}\cup T_j, j=1,2,\cdots,n-1, \delta(t_0)=[t_0,t_0+\delta]\subset T_0, \delta(t_n)=[t_n-\delta,t_n]\subset T_{n-1}$ and $\delta(t_j)\cap\delta(t_k)=\emptyset, j\neq k$, and the set $X_i=\{x_1^{(i)},x_2^{(i)},\cdots\}\subset\delta(t_i)$ for $i=0,1,2,\cdots,n$ is chosen so that $x_1^{(i)}=t_i$ and the elements in X_i do not have to be distinct from each other. Let

$$\omega_k^{(i)}=\omega_k^{(i)}(x)=(x-x_1^{(i)})(x-x_2^{(i)})\cdots(x-x_k^{(i)}), \tag{1.1}$$

and the following interpolation conditions by given by means of divided difference of smooth function $f(x)$

$$y_m^{(i)}=f[x_1^{(i)},x_2^{(i)},\cdots,x_{m+1}^{(i)}],\quad m=0,1,\cdots,k-1; i=0,1,\cdots,n. \tag{1.2}$$

Definition 1 *Real function $R(x)$ in $[a,b]$ is called generalized rational spline (GRS) of type (r,l) with respect to $f(x)$ if it satisfies the following conditions*

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(a) *For each subinterval* $T_i, R(x) = P_i(x)/Q_i(x), i = 0, 1, \cdots, n-1$ *with* $P_i(x) \in H_r, Q_i(x) \in H_l$, *here* H_k *denotes the collection of all polynomials of degree* k.

(b) $\sum_{m=0}^{k-1} y_m^{(i)} \omega_m^{(i)}(x) - P_i(x)/Q_i(x) = O(\omega_k^{(i)}(x))$.

(c) $\sum_{m=0}^{k-1} y_m^{(i+1)} \omega_m^{(i+1)}(x) - P_i(x)/Q_i(x) = O(\omega_k^{(i+1)}(x))$

(d) $r + l = 2k - 1$.

It follows from the above definition

$$P_{i+1}(x)/Q_{i+1}(x) - P_i(x)/Q_i(x) = O(\omega_k^{(i+1)}(x)), \tag{1.3}$$

where by $O(\omega_k^{(i)}(x))$ we mean $d_k\omega_k^{(i)}(x) + d_{k+1}\omega_{k+1}^{(i)}(x) + \cdots$. If in X_{i+1} appear $M_{i+1}(\leq k)$ coincident elements, say, $t_{i+1} = x_1^{(i+1)} = x_2^{(i+1)} = \cdots = x_{M_{i+1}}^{(i+1)}$, then we know from (1.3) that $R(x)$ is a rational spline function with smoothness orders $M_{i+1} - 1$ over $T_i \cup T_{i+1}$. Let δ tend to zero, then the GRS $R(x)$ turns out to be the Padé spline the authors first introduced and studied (see [9,10]).

Very few results have been achieved in interpolating rational splines since R.Schaback ([5]) pioneered the study of this subject in early seventies, though constant efforts have been made to develop numerical methods leading to the constructions of rational splines for many years. There is no doubt that the biggest obstacle against the advance of rational splines results from their nonlinearity. In Schaback's case, the determination of rational splines are subject to a system of nonlinear equations which are difficult to carry out. To make up for this, R.H.Wang et al. ([6,7]) concretely studied a few special types of interpolating rational splines consisting of both polynomial parts and rational parts which are convenient for computation due to their linear determinability. However we observe that there is a shortage in [6] for lack of numerical recursive algorithms.

The object of this paper is to present a few effective approaches to the construction of generalized rational splines in terms of Newton's interpolation. An outline of the contents is roughly arranged as follows. In Section 2 we construct a few kinds of GRSs by various methods, including Thiele's reciprocal difference method, Padé approximant method, generalized Q.D.algorithm and generalized ε-algorithm. In Section 3 we establish in direct way the determinant representations and remainder expressions of GRSs with type $(k-1, k)$, and in Section 4 we give some properties of them.

2. Costructions of GRSs

Before stating our results, we need the following propositions.

Proposition 2.1 *The conditions (b) and (c) in Definition 1 are equivalent to the requirements*

$$P_i(x_j^{(i)})/Q_i(x_j^{(i)}) = f(x_j^{(i)}), P_i(x_j^{(i+1)})/Q_i(x_j^{(i+1)}) = f(x_j^{(i+1)}), \quad j = 1, 2, \cdots, k.$$

For the sake of convenience, we introduce the following notations

$$x_s = x_s^{(i)}, \quad s = 1, 2, \cdots, k, \tag{2.1}$$

$$x_s = x_{s-k}^{(i+1)}, \quad s = k+1, k+2, \cdots, 2k, \tag{2.2}$$

$$\psi_m(x) = (x - x_1)(x - x_2)\cdots(x - x_m), \quad m \le 2k. \tag{2.3}$$

Construction 1 of GRSs

Initialisation. Define

$$b_1 = f(x_1), \tag{2.4}$$

and, for $j = 2, 3, \cdots, 2k$,

$$R^{(2)}(x_j) = \frac{x_j - x_1}{f(x_j) - b_1} \tag{2.5}$$

Iteration. For $s = 2, 3, \cdots, 2k - 1$, define

$$b_s = R^{(s)}(x_s), \tag{2.6}$$

and for $j = s + 1, s + 2, \cdots, 2k$,

$$R^{(s+1)}(x_j) = \frac{x_j - x_s}{R^{(s)}(x_j) - b_s} \tag{2.7}$$

Termination. Define

$$b_{2k} = R^{(2k)}(x_{2k}). \tag{2.8}$$

The resulting construct is

$$R(x) = b_1 + \frac{x - x_1}{b_2} + \frac{x - x_2}{b_3} + \cdots + \frac{x - x_{2k-1}}{b_{2k}} \tag{2.9}$$

We point out that the above continued fraction $R(x)$ can serve as the $P_i(x)/Q_i(x)$ in Definition 1 with $P_i(x)$ and $Q_i(x)$ belonging to H_k and H_{k-1} respectively. In fact the iterative steps (2.4)–(2.8) is based on Proposition 2.1 and the following considerations. Let

$$R^{(s)}(x) = b_s + \frac{x - x_s}{b_{s+1}} + \cdots + \frac{x - x_{2k-1}}{b_{2k}}, \quad s = 1, 2, \cdots, 2k, \tag{2.10}$$

then

$$R^{(s)}(x) = b_s + \frac{x - x_s}{R^{(s+1)}(x)}, \tag{2.11}$$

and from (2.11) follow (2.6)–(2.8). It is easy to see that $R(x) = R^{(1)}(x)$ is a rational function of type $(k, k-1)$. In order to make $R(x)$ serve as $P_i(x)/Q_i(x)$, we are obliged to set stages (2.4) and (2.5) by menas of (2.11).

If $g(x) = 1/f(x)$ is defined on the union $\delta(t_0) \cup \delta(t_1) \cup \cdots \cup \delta(t_n)$, then we can obtain a GRS of type $(k-1, k)$ through the construction below

Construction 2

Define

$$R(x) = \frac{1}{b_1} + \frac{x - x_1}{b_2} + \cdots + \frac{x - x_{2k-1}}{b_{2k}}, \tag{2.12}$$

where $b_s, s = 1, 2, \cdots, 2k$ can be computed by the same iterative procedure as in Construction 1 except for that $f(x)$ is replaced, with its inverse function $g(x)$.

Construction 3

Define

$$R(x) = \sum_{m=0}^{k-1} g_m^{(i)} \omega_m^{(i)}(x) + \omega_k^{(i)}(x) R^*(x), \tag{2.13}$$

where

$$R^*(x) = b_{k+1} + \frac{x - x_{k+1}}{b_{k+2}} \underset{+}{} \cdots \underset{+}{} \frac{x - x_{2k-1}}{b_{2k}}, \tag{2.14}$$

or define

$$R(x) = \sum_{m=0}^{k-1} y_m^{(i+1)} \omega_m^{(i+1)}(x) = \omega_k^{(i+1)}(x) \bar{R}(x), \tag{2.15}$$

where

$$\bar{R}(x) = b_1 + \frac{x - x_1}{b_2} \underset{+}{} \cdots \underset{+}{} \frac{x - x_{k-1}}{b_k}. \tag{2.16}$$

By Newton's interpolation formula, we can write

$$f(x) = \sum_{m=0}^{k-1} y_m^{(i)} \omega_m^{(i)}(x) + \omega_k^{(i)}(x) f^*(x), \tag{2.17}$$

or

$$f(x) = \sum_{m=0}^{k-1} y_m^{(i+1)} \omega_m^{(i+1)}(x) + \omega_k^{(i+1)}(x) \bar{f}(x). \tag{2.18}$$

We compute $R^*(x)$ and $\bar{R}(x)$ in a similar manner to (2.4)–(2.9) so that $f^*(x_s) = R^*(x_s)$ for $s = k+1, \cdots, 2k$ and $\bar{f}(x_s) = \bar{R}(x_s)$ for $s = 1, 2, \cdots, k$. At this time, there always hold $R(x_s) = f(x_s)$ for $s = 1, 2, \cdots, 2k$. Therefore by Proposition 2.1, $R(x)$ defined in (2.13) and (2.15) plays the part $P_i(x)/Q_i(x)$ of the GRS of type $(k + [k/2], [k - 1/2])$, where $[x]$ stands for the largest integer not exceeding x.

Remark 1 Changing the continued fraction $R^*(x)$ in (2.13) or $\bar{R}(x)$ in (2.15) into the form analogous to (2.12) and using the algorithm therein, one can obtain a GRS of type $(k + [k - 1/2], [k/2])$.

Construction 4

Define

$$b_s = f(x_s), \quad s = 1, 2, \cdots, 2k, \tag{2.19}$$

$$b_{s,t} = \frac{x_s - x_t}{b_s - b_t}, \tag{2.20}$$

and in general for $j \geq 2$,

$$b_{1,2,\cdots,j} = \frac{x_j - x_{j-1}}{b_{1,\cdots,j-2,j} - b_{1,\cdots,j-1}} \tag{2.21}$$

Then a GRS of type $(k, k-1)$ is given by

$$R(x) = b_1 + \frac{x - x_1}{b_{1,2}} \underset{+}{} \cdots \underset{+}{} \frac{x - x_{2k-1}}{b_{1,2,\cdots,2k}} \tag{2.22}$$

It suffices to show $R(x_s) = f(x_s)$ for $s = 1, 2, \cdots, 2k$. From (2.22) and (2.19)–(2.21) we have

$$\begin{aligned} R(x_s) &= b_1 + \frac{x_s - x_1}{b_{1,2}} + \cdots + \frac{x_s - x_{s-1}}{b_{1,2,\cdots,s}} \\ &= b_1 + \frac{x_s - x_1}{b_{1,2}} + \cdots + \frac{x_s - x_{s-2}}{b_{1,2,\cdots,s-2,s}} \\ &= \cdots \\ &= b_1 + \frac{x_s - x_1}{b_{1,s}} = b_s = f(x_s). \end{aligned}$$

Remark 2 Construction 4 is practically based on Thiele's reciprocal difference method. If $f(x_s) \neq 0$ for $s = 1, 2, \cdots, 2k$, then by letting $b_s = 1/f(x_s)$ and keeping (2.20) and (2.21) unchanged one gets a GRS $1/R(x)$ of type $(k-1, k)$, where $R(x)$ is given by (2.22). One can also apply Construction 4 to Construction 3 to yield other types of GRSs.

Construction 5

Set

$$f_{s,t} = f[x_s, x_{s+1}, \cdots, x_t], \tag{2.23}$$

$$P_i(x) = \begin{vmatrix} f_{l+1,r+2} & \cdots & f_{l+1,r+l+1} & \sum_{j=l+1}^{r+1} f_{l+1,j}\psi_{j-1}(x) \\ f_{l,r+2} & \cdots & f_{l,r+l+1} & \sum_{j=l}^{r+1} f_{l,j}\psi_{j-1}(x) \\ \vdots & \ddots & \vdots & \vdots \\ f_{1,r+2} & \cdots & f_{1,r+l+1} & \sum_{j=1}^{r+1} f_{1,j}\psi_{j-1}(x) \end{vmatrix}, \tag{2.24}$$

$$Q_i(x) = \begin{vmatrix} f_{l+1,r+2} & \cdots & f_{l+1,r+l+1} & \psi_l(x) \\ f_{l,r+2} & \cdots & f_{l,r+l+1} & \psi_{l-1}(x) \\ \vdots & \ddots & \vdots & \vdots \\ f_{1,r+2} & \cdots & f_{1,r+l+1} & 1 \end{vmatrix}, \tag{2.25}$$

where $f_{s,t} = 0$ if $s > t$. It is not difficult to verify that $P_i(x)/Q_i(x)$ will be in conformity with the requirements in Definition 1 and act as a GRS of type (r, l) provided $Q_i(x_s) \neq 0$ for $s = 1, 2, \cdots, 2k$ and $r + l = 2k - 1$.

Keeping the notations introduced in Construction 3, we have the following

Construction 6

Let

$$P_i(x)/Q_i(x) = \sum_{m=0}^{k-1} y_m^{(i)}\omega_m^{(i)}(x) + \omega_k^{(i)}(x)P_i^*(x)/Q_i^*(x), \tag{2.26}$$

where $P_i^*(x)$ and $Q_i^*(x)$ are obtained through substituting $f_{s,t}^* = f^*[x_{s+k}, x_{s+k+1}, \cdots, x_{t+k}]$ and $\omega_j^{(i+1)}(x)$ into (2.24),(2.25) for $f_{s,t}$ and $\psi_j(x)$ respectively. By Construction 3 and

Construction 5, $P_i(x)/Q_i(x)$ in (2.26) provides us with a GRS of type $(\max(k+r,k+l-1),l)$, here $r+l=k-1$. In order to maintain the condition (d) in Definition 1, it is necessary to require $l \le [k/2]$. If one defines

$$P_i(x)/Q_i(x) = \sum_{m=0}^{k-1} y_m^{(i+1)}\omega_m^{(i+1)}(x) + \omega_k^{(i+1)}(x)\bar{P}_i(x)/\bar{Q}(x), \tag{2.27}$$

where $\bar{P}_i(x)$ and $\bar{Q}_i(x)$ come into being by making the substitution $\bar{f}_{s,t} = \bar{f}[x_s, x_{s+1}, \cdots, x_t]$ for $f_{s,t}$ in (2.24) and (2.25), and by restricting $r+l=k-1$ as well as $1 \le [k/2]$, then $P_i(x)/Q_i(x)$ in (2.27) turns out to be a GRS of type $(k+r,l)$ in the light of Definition 1.

Remark 3 We mention by the way that the GRS defined in (2.26) is also of type $(k+r,l)$, which is due to the restrictions $r+l=k-1, l \le [k/2]$ and the fact $[k/2]+[k-1/2]=k-1$. The following construction is based on the generalized Q.D.algorithm designed by Wuytack[8] and Graves- Morris[3] (see also [1]).

Construction 7

Let us introduce a notation with wider meaning than one in (2.3) as follows

$$\psi_{m,n}(x) = (x-x_m)(x-x_{m+1})\cdots(x-x_{n-1}), \tag{2.28}$$

from which yield $\psi_s(x)$ if $m=1$ and $n=s+1, \omega_s^{(i)}(x)$ if $s \le k, m=1$ and $n=s+1$, and $\omega_s^{(i+1)}(x)$ if $s \le k, m=k+1$ and $n=k+s+1$. A function $g(x)$ can be expanded into the following form

$$g(x) = g_{s,s} + g_{s,s+1}\psi_{s,s+1}(x) + \cdots g_{s,t}\psi_{s,t}(x) + O(\psi_{s,t+1}(x)), \tag{2.29}$$

where

$$g_{s,t} = g[x_s, x_{s+1}, \cdots, x_t]. \tag{2.30}$$

Let

$$R_{s,t}(g;x) = \frac{g_{s,s}}{1} - \frac{q_1^s(x-x_s)}{1} - \frac{e_1^s(x-x_{s+1})}{1} - \frac{q_2^s(x-x_{s+2})}{1} - \frac{e_2^s(x-x_{s+3})}{1} - \cdots - \frac{q^s_{[t-s/2]}(x-x_{s+2[t-s/2]-2})}{1} - \frac{e^s_{[t-s/2]}(x-x_{s+2[t-s/2]-2})}{1} - \frac{v(t-s)q^s_{[t-s+1/2]}(x-x_{s+2[t-s/2]})}{1}, \tag{2.31}$$

where $v(t-s)=0$ if $t-s$ is an even number, and $v(t-s)=1$ if $t-s$ is an odd number. The above rational function of type $([t-s/2],[t-s+1/2])$ can be determined by deriving its coefficients from the following algorithm:

Initialization For $j=s,s+1,t-1$, define

$$x_1^j = x_{j+1} - x_j, \tag{2.32}$$

$$e_0^{j+1} = 0, \tag{2.33}$$

$$q_1^j = (x_1^j + g_{s,j}/g_{s,j+1})^{-1}, \tag{2.34}$$
$$e_1^j = -q_1^j - q_1^{j+1}(q_1^j x_1^j - 1). \tag{2.35}$$

Recurrence For $j = s, s+1, \cdots$ and $m = 2, 3, \cdots, [t-s+1/2]$, we construct all well-defined quantities q_m^j, e_m^j recursively from the formulas

$$x_m^j = x_{j+2m-1} - x_{j+2m-2}, \tag{2.36}$$
$$q_m^j = (x_m^j - \frac{e_{m-1}^j(q_{m-1}^{j+1} + e_{m-2}^{j+1})(x_m^j e_{m-1}^{j+1} - 1)}{(e_{m-1}^j + q_{m-1}^j)q_{m-1}^{j+1} e_{m-1}^{j+1}})^{-1}, \tag{2.37}$$
$$e_m^j = -q_m^j + (x_m^j q_m^j - 1)(e_{m-1}^{j+1} + q_m^{j+1})(x_m^j e_{m-1}^{j+1} - 1)^{-1}. \tag{2.38}$$

The algorithm decribed above aims at obtaining the relations

$$R_{s,t}(g; x_n) = g(x_n), \quad n = s, s+1, \cdots, t, \tag{2.39}$$

and hence enables one, in virtue of (2.17) and (2.18), to construct three kinds of GRSs as follows

(A) $P_i(x)/Q_i(x) = R_{1,2k}(f; x)$,

(B) $P_i(x)/Q_i(x) = \sum_{m=0}^{k-1} f_{1,m+1}\psi_{1,m+1}(x) + \psi_{1,k+1}(x)R_{k+1,2k}(f^*; x)$,

(C) $P_i(x)/Q_i(x) = \sum_{m=0}^{k-1} f_{k+1,k+m+1}\psi_{k+1,k+m+1}(x) + \psi_{k+1,2k+1}(x)R_{1,k}(\bar{f}; x)$.

It is easy to see that (A) is a GRS of type $(k-1, k)$ while both (B) and (C) are GRss of type $(k + [k-1/2], [k/2])$.

Remark 4 Algorithms for rational interpolation, in general, may be classified into those designed to solve the coefficient problem and those designed to solve the value problem. For example, the generalized Q.D.algorithm described in Construction 7 is for the purpose of solving the coefficient problem in the continued fraction of form (2.31). The generalized ε-algorithm ([2]) is a method of entailing the evaluation of $P_i(x)/Q_i(x)$ in Definition 1 at some prespecified value of x without seeking for the explicit representations of its coefficients such as in (2.24) and (2.25). This algorithm is based on the formal identity

$$(x - x_{s+t+1})(\varepsilon_{s+1}^{(t)} - \varepsilon_{s-1}^{(t+1)})(\varepsilon_s^{(t+1)} - \varepsilon_s^{(t)}) = 1 \tag{2.40}$$

for indices s, t in the range $s = 0, 1, 2, \cdots$ and $t \geq -[s/2]$. The initialization conditions are

$$\varepsilon_{-1}^{(t)} = 0, \quad t = 0, 1, 2, \cdots, \tag{2.41}$$
$$\varepsilon_{2s}^{(-s-1)} = 0, \quad s = 0, 1, 2, \cdots, \tag{2.42}$$
$$\varepsilon_0^{(t)} = \sum_{j=1}^{t+1} f_{1,j}\psi_{1,j}(x). \tag{2.43}$$

Consequently a rational interpolant of type $(s+t, s)$ fitting $f(x)$ at x_j for $j = 1, 2, \cdots, 2s+t+1$ is given by $\varepsilon_{2s}^{(t)}$, from which are deduced the GRSs $\varepsilon_{2k-2u}^{(2u-1)}$ of types $(k+u-1, k-u)$ with u ranging from $1-k$ to k.

3. Direct Construction of the GRS of Type $(k-1, k)$

We adopt the notations previously introduced without further declaration. Let

$$s_{k,i}(x) = \sum_{m=0}^{k-1} y_m^{(i)} \psi_{1,m+1}(x), \tag{3.1}$$

$$P_i(x) = p_0 + p_1\psi_{1,2}(x) + \cdots + p_{k-1}\psi_{1,k}(x) \tag{3.2}$$

$$= \bar{p}_0 + \bar{p}_1\psi_{k+1,k+2}(x) + \cdots + \bar{p}_{k-1}\psi_{k+1,2k}(x), \tag{3.3}$$

$$Q_i(x) = q_0 + q_1\psi_{1,2}(x) + \cdots + q_k\psi_{1,k+1}(x) \tag{3.4}$$

$$= \bar{q}_0 + \bar{q}_1\psi_{k+1,k+2}(x) + \cdots + \bar{q}_k\psi_{k+1,2k+1}(x), \tag{3.5}$$

and denote by Z_i the collection of all zeros of $Q_i(x)$. If $Z_i \cap (X_i \cup X_{i+1}) = \emptyset$, then it follows from (3.1)–(3.5) that the condition (b) in Definition 1 is equivalent to

$$\sum_{t=0}^{k} s_{k,i}(x) q_t \psi_{1,t+1}(x) - \sum_{t=0}^{k-1} p_t \psi_{1,t+1}(x) = \sum_{t=k}^{2k-1} r_t^{(i)} \psi_{1,t+1}(x). \tag{3.6}$$

Let

$$s_{m,n}^{(i)} = s_{k,i}[x_m, \cdots, x_n], \quad n \geq m \tag{3.7}$$

and $s_{m,n}^{(i)} = 0$ for $n < m$, we have

$$s_{k,i}(x)\psi_{1,t+1}(x) = \sum_{j=1}^{k} s_{t+1,t+j}^{(i)} \psi_{1,t+j}(x). \tag{3.8}$$

Substituting (3.8) into (3.6) yields

$$\sum_{m=0}^{k-1} \psi_{1,m+1} \sum_{j=0}^{m} s_{j+1,m+1}^{(i)} q_j + \sum_{m=k}^{2k-1} \psi_{1,m+1} \sum_{j=m-k+1}^{m} s_{j+1,m+1}^{(i)} q_j$$
$$= \sum_{m=0}^{k-1} p_m \psi_{1,m+1} + \sum_{m=k}^{2k-1} r_m^{(i)} \psi_{1,m+1}. \tag{3.9}$$

It follows from (3.9)

$$p_m = \sum_{j=0}^{m} s_{j+1,m+1}^{(i)} q_j, \quad m = 0, 1, \cdots, k-1, \tag{3.10}$$

$$r_m^{(i)} = \sum_{j=m-k+1}^{k} s_{j+1,m+1}^{(i)} q_j, \quad m = k, k+1, \cdots, 2k-1. \tag{3.11}$$

Similarly we can get from the condition (c) in Definition 1

$$\bar{p}_m = \sum_{j=0}^{m} s_{j+1,m+1}^{(i+1)} \bar{q}_j, \quad m = 0, 1, \cdots, k-1, \tag{3.12}$$

$$\bar{r}_m^{(i)} = \sum_{j=m-k+1}^{k} s_{j+1,m+1}^{(i+1)} \bar{q}_j, \quad m = k, k+1, \cdots, 2k-1, \tag{3.13}$$

where $\bar{r}_m^{(i)}, m = k, k+1, \cdots, 2k-1$ satisfy the following relation

$$\sum_{t=0}^{k} s_{k,i+1}(x) \bar{q}_t \psi_{k+1,k+t+1}(x) - \sum_{t=0}^{k-1} \bar{p}_t \psi_{k+1,k+t+1}(x) = \sum_{t=k}^{2k-1} \bar{r}_t^{(i)} \psi_{k+1,k+t+1}(x). \tag{3.14}$$

From (3.2)–(3.5) we have

$$\bar{p}_m = \sum_{t=m}^{k-1} p_t \psi_{1,t+1}[x_{k+1}, \cdots, x_{m+k+1}], \tag{3.15}$$

$$\bar{q}_m = \sum_{t=m}^{k} q_t \psi_{1,t+1}[x_{k+1}, \cdots, x_{m+k+1}], \tag{3.16}$$

Putting (3.10),(3.15) and (3.16) into (3.12) results in

$$\sum_{j=0}^{k} v_{m,j} q_j = 0, \quad m = 0, 1, \cdots, k-1, \tag{3.17}$$

where

$$v_{m,j} = \begin{cases} \displaystyle\sum_{t=0}^{k-m-1} s_{j+1,m+t+1}^{(i)} \psi_{1,m+t+1}[x_{k+1}, \cdots, x_{k+m+1}] \\ \displaystyle\quad - \sum_{t=1}^{j+1} s_{t,m+1}^{(i+1)} \psi_{1,j+1}[x_{k+1}, \cdots, x_{k+t}], & 0 \le j \le m, \\ \displaystyle\sum_{t=j}^{k-1} s_{j+1,t+1}^{(i)} \psi_{1,t+1}[x_{k+1}, \cdots, x_{k+m+1}] \\ \displaystyle\quad - \sum_{t=1}^{m+1} s_{t,m+1}^{(i+1)} \psi_{1,j+1}[x_{k+1}, \cdots, x_{k+t}], & m+1 \le j \le k-1, \\ \displaystyle -\sum_{t=1}^{m+1} s_{t,m+1}^{(i+1)} \psi_{1,k+1}[x_{k+1}, \cdots, x_{k+t}], & j = k. \end{cases} \tag{3.18}$$

Apart from a common factor, we can derive from (3.17)

$$Q_i(x) = \begin{vmatrix} 1 & \psi_{1,2}(x) & \cdots & \psi_{1,k+1}(x) \\ v_{0,0} & v_{0,1} & \cdots & v_{0,k} \\ \vdots & \vdots & \ddots & \vdots \\ v_{k-1,0} & v_{k-1,1} & \cdots & v_{k-1,k} \end{vmatrix} \tag{3.19}$$

$$P_i(x) = \begin{vmatrix} \sum_{m=0}^{k-1} s_{1,m+1}^{(i)}\psi_{1,m+1}(x) & \cdots & \sum_{m=k-1}^{k-1} s_{k,m+1}^{(i)}\psi_{1,m+1}(x) & 0 \\ v_{0,0} & \cdots & v_{0,k-1} & v_{0,k} \\ \vdots & \ddots & \vdots & \vdots \\ v_{k-1,0} & \cdots & v_{k-1,k-1} & v_{k-1,k} \end{vmatrix} \tag{3.20}$$

As a result of comparing (3.11) with (3.19), there holds

$$r_m^{(i)} = \begin{vmatrix} 0 & \cdots & 0 & s_{m-k+2,m+1}^{(i)} & \cdots & s_{k+1,m+1}^{(i)} \\ v_{0,0} & \cdots & v_{0,m-k} & v_{0,m-k+1} & \cdots & v_{0,k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_{k-1,0} & \cdots & v_{k-1,m-k} & v_{k-1,m-k+1} & \cdots & v_{k-1,k} \end{vmatrix} \tag{3.21}$$

Substituting (3.16) into (3.13) and setting

$$H_{j,m} = \sum_{t=m-k+2}^{j+1} s_{t,m+1}^{(i+1)}\psi_{1,j+1}[x_{k+1},\cdots,x_{k+t}], \tag{3.22}$$

we obtain

$$\bar{r}_m^{(i)} = \begin{vmatrix} 0 & \cdots & 0 & H_{m-k+1,m} & \cdots & H_{k,m} \\ v_{0,0} & \cdots & v_{0,m-k} & v_{0,m-k+1} & \cdots & v_{0,k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_{k-1,0} & \cdots & v_{k-1,m-k} & v_{k-1,m-k+1} & \cdots & v_{k-1,k} \end{vmatrix}. \tag{3.23}$$

Starting from (3.3) and (3.5), one may derive

$$Q_i(x) = \begin{vmatrix} 1 & \psi_{k+1,k+2}(x) & \cdots & \psi_{k+1,2k+1}(x) \\ \bar{v}_{0,0} & \bar{v}_{0,1} & \cdots & \bar{v}_{0,k} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{v}_{k-1,0} & \bar{v}_{k-1,1} & \cdots & \bar{v}_{k-1,k} \end{vmatrix}, \tag{3.24}$$

$$P_i(x) = \begin{vmatrix} \sum_{m=0}^{k-1} s_{1,m+1}^{(i+1)}\psi_{k+1,k+m+1}(x) & \cdots & \sum_{m=k-1}^{k-1} s_{k,m+1}^{(i+1)}\psi_{k+1,k+m+1}(x) & 0 \\ \bar{v}_{0,0} & \cdots & \bar{v}_{0,k-1} & \bar{v}_{0,k} \\ \vdots & \ddots & \vdots & \vdots \\ \bar{v}_{k-1,0} & \cdots & \bar{v}_{k-1,k-1} & \bar{v}_{k-1,k} \end{vmatrix} \tag{3.25}$$

$$r_m^{(i)} = \begin{vmatrix} 0 & \cdots & 0 & \bar{H}_{m-k+1,m} & \cdots & \bar{H}_{k,m} \\ \bar{v}_{0,0} & \cdots & \bar{v}_{0,m-k} & \bar{v}_{0,m-k+1} & \cdots & \bar{v}_{0,k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \bar{v}_{k-1,0} & \cdots & \bar{v}_{k-1,m-k} & \bar{v}_{k-1,m-k+1} & \cdots & \bar{v}_{k-1,k} \end{vmatrix}. \tag{3.26}$$

$$\bar{r}_m^{(i)} = \begin{vmatrix} 0 & \cdots & 0 & s_{m-k+2,m+1}^{(i+1)} & \cdots & s_{k+1,m+1}^{(i+1)} \\ \bar{v}_{0,0} & \cdots & \bar{v}_{0,m-k} & \bar{v}_{0,m-k+1} & \cdots & \bar{v}_{0,k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \bar{v}_{k-1,0} & \cdots & \bar{v}_{k-1,m-k} & \bar{v}_{k-1,m-k+1} & \cdots & \bar{v}_{k-1,k} \end{vmatrix}, \tag{3.27}$$

where

$$\bar{v}_{m,j} = \begin{cases} \sum_{t=0}^{k-m-1} s_{j+1,m+t+1}^{(i+1)} \psi_{k+1,k+m+t+1}[x_1, \cdots, x_{m+1}] \\ \quad - \sum_{t=1}^{j+1} s_{t,m+1}^{(i)} \psi_{k+1,k+j+1}[x_1, \cdots, x_t], & 0 \le j \le m, \\ \sum_{t=j}^{k-1} s_{j+1,t+1}^{(i+1)} \psi_{k+1,k+t+1}[x_1, \cdots, x_{m+1}] \\ \quad - \sum_{t=1}^{m+1} s_{t,m+1}^{(i)} \psi_{k+1,k+j+1}[x_1, \cdots, x_t], & m+1 \le j \le k-1, \\ - \sum_{t=1}^{m+1} s_{t,m+1}^{(i)} \psi_{k+1,2k+1}[x_1, \cdots, x_t], & j = k. \end{cases} \tag{3.28}$$

$$\bar{H}_{j,m} = \sum_{t=m-k+2}^{j+1} s_{t,m+1}^{(i)} \psi_{k+1,k+j+1}[x_1, \cdots, x_t]. \tag{3.29}$$

Notice that the $Q_i(x)$ in (3.19) and $P_i(x)$ in (3.20) are, in general, different from those in (3.24) and (3.25), nevertheless the ratios $P_i(x)/Q_i(x)$ are the very same. If $Z_i \cap (X_i \cup X_{i+1}) = \emptyset$ for $i = 0, 1, \cdots, n-1$, then it follows form (3.6) and (3.14)

$$\frac{P_{i+1}(x)}{Q_{i+1}(x)} - \frac{P_i(x)}{Q_i(x)} = \sum_{t=k}^{2k-1} \left(\frac{\bar{r}_t^{(i)}}{Q_i(x)} - \frac{r_t^{(i+1)}}{Q_{i+1}(x)}\right) \psi_{k+1,k+t+1}(x) \tag{3.30}$$

and hence we obtain a general representation for the GRS of type $(k-1, k)$ and follows

$$R(x) = \frac{P_0(x)}{Q_0(x)} + \sum_{i=0}^{n-2} \sum_{t=k}^{2k-1} \left(\frac{\bar{r}_t^{(i)}}{Q_i(x)} - \frac{r_t^{(i+1)}}{Q_{i+1}(x)}\right) \psi_{k+1,k+t+1}(x)_+, \tag{3.31}$$

where

$$\psi_{k+1,k+t+1}(x)_+ = \begin{cases} \psi_{k+1,k+t+1}(x), & \text{if } x \in \cup_{j \ge t+1} T_j, \\ 0, & \text{if } x \in \cup_{j < i+1} T_j. \end{cases} \tag{3.32}$$

4. Some Properties of GRSs

With the notation in the preceding sections, we have

Theorem 4.1 *Any two GRSs of the same type (r,l) are equal.*

Proof Assume that $R(x)$ and $R^*(x)$ are two GRSs of the same type (r,l) and their expressions restricted on the subinterval $T_i(i=0,1,\cdots,n-1)$ are $P_i(x)/Q_i(x)$ and $P_i^*(x)/Q_i^*(x)$ respectively. It suffices to verify that $P_i(x)/Q_i(x)$ is equal to $P_i^*(x)/Q_i^*(x)$. It follows from Definition 1.

$$\frac{P_i(x)}{Q_i(x)}-\frac{P_i^*(x)}{Q_i^*(x)}=O(\psi_{1,k+1}(x)),$$
$$\frac{P_i(x)}{Q_i(x)}-\frac{P_i^*(x)}{Q_i^*(x)}=O(\psi_{k+1,2k+1}(x)).$$

Let

$$U_i(x)=P_i(x)Q_i^*(x)-P_i^*(x)Q_i(x), \tag{4.1}$$

then $U_i(x)$ is a polynomial of degree not exceeding $2k-1$ satisfying

$$U_i(x)=O(\psi_{1,k+1}(x)), \tag{4.2}$$

$$U_i(x)=O(\psi_{k+1,2k+1}(x)), \tag{4.3}$$

therefore

$$W_{ik}(x)=U_i(x)/\psi_{1,k+1}(x) \tag{4.4}$$

is a polynomial of degree at most $k-1$. From (4.3) follows $W_{ik}(x_j)=0$ for $j=k+1,\cdots,2k$, which implies $W_{ik}(x)\equiv 0$ and hence $U_i(x)\equiv 0$, as asserted.

Theorem 4.2 *There hold the following determinant identities with respect to the GRS of type $(k-1,k)$*

$$\begin{vmatrix} 1 & \psi_{1,2}(x) & \cdots & \psi_{1,k+1}(x) \\ v_{0,0} & v_{0,1} & \cdots & v_{0,k} \\ \vdots & \vdots & \ddots & \vdots \\ v_{k-1,0} & v_{k-1,1} & \cdots & v_{k-1,k} \end{vmatrix} \cdot \begin{vmatrix} \bar v_{0,0} & \cdots & \bar v_{0,k-1} \\ \vdots & \ddots & \vdots \\ \bar v_{k-1,0} & \cdots & \bar v_{k-1,k-1} \end{vmatrix}$$
$$= \begin{vmatrix} 1 & \psi_{k+1,k+2}(x) & \cdots & \psi_{k+1,2k+1}(x) \\ \bar v_{0,0} & \bar v_{0,1} & \cdots & \bar v_{0,k} \\ \vdots & \vdots & \ddots & \vdots \\ \bar v_{k-1,0} & \bar v_{k-1,1} & \cdots & \bar v_{k-1,k} \end{vmatrix} \cdot \begin{vmatrix} v_{0,0} & \cdots & v_{0,k-1} \\ \vdots & \ddots & \vdots \\ v_{k-1,0} & \cdots & v_{k-1,k-1} \end{vmatrix}$$

Proof The proof is completed by normalizing the $Q_i(x)$ in (3.19) and in (3.24) to be monic polynomials respectively and then making them equalized.

Theorem 4.3 *Denote $R(f;x)=R(x), P_i(f;x)=P_i(x)$ and $Q_i(f;x)=Q_i(x)$, then there holds $R(cf;x)=cR(f;x)$ for any constant c, where $R(x)$ is a GRS of type $(k-1,k)$.*

Proof If $c = 0$, the conclusion is immediately drawn from Definition 1 since on this occasion $P_i(cf;x) = O(\psi_{1,k+1}(x))$, which implies $P_i(cf;x) = 0$, hence $R(cf;x) = 0$. If $c \neq 0$, then it follows from (3.18)–(3.20) that $P_i(cf;x) = c^{k+1}P_i(f;x), Q_i(cf;x) = c^k Q_i(f;x)$ and thereby $R(cf;x) = cR(f;x)$.

There is no doubt that the above defined $R(f;x)$ can be regarded as a nonlinear operator with respect to the function $f(x)$, which satisfies the following continuity theorem.

Theroem 4.4 *For fixed positive integers k and n, the generalized rational spline operator $R(f;x)$ of type $(k-1,k)$ is continuous in regard of f on any compact subset X in $[a,b]$ excluding all possible zeros of $\prod_{i=0}^{n-1} Q_i(x)$, i.e., $\|f-f^*\| \to 0$ implies $\|R(f;\cdot)-R(f^*;\cdot)\|_x \to 0$, where $\|f\| = \max_{0\le i<n-1}\sum_{m=1}^{k} f[x_1,x_2,\cdots,x_m]$ and $\|F\|_X = \max_{t\in X}|F(t)|$.*

Proof Since $[a,b] = \cup_{i=0}^{n-1}T_i, X \subset [a,b]$, we have $X = \cup_{i=0}^{n-1}\bar{X}_i$ and $\|F\|_X = \max_{0\le i\le n-1}\|F\|_{\bar{X}_i}$ where $\bar{X}_i = X \cap T_i, i = 0,1,\cdots,n-1$. Notice that the $v_{m,j}$ defined in (3.18) are actually linear functionals regarding f, which we denote by $v_{m,j}(f)$. It is easy to know from (3.18) that for $0 \le m \le k-1, 0 \le j \le k$ and $0 \le i \le n-1$, there exists an absolute positive constant C such that $|v_{m,j}(f)| \le C\|f\|$ for all indices m,j and i within given ranges. Therefore $\|f-f^*\| \to 0$ leads to $v_{m,j}(f) \to v_{m,j}(f^*)$ for $0 \le m \le k-1, 0 \le j \le k$ and $0 \le i \le n-1$, which ensures $Q_i(f;x) \to Q_i(f^*;x)$ and $P_i(f;x) \to P_i(f^*;x)$ by (3.19) and (3.20). The fact that $Q_i(f;x)$ has no zero on $\bar{X}_i$ shows that there exists constant $d_i > 0$ such that $\inf_{x\in\bar{X}_i}|Q_i(f;x)| = d_i$ and $\inf_{x\in\bar{X}}|Q_i(f^*;x)| \ge d_i/2$ for f^* sufficiently close to f. Thus it follows from $\|f-f^*\| \to 0$ and boundness of $\|Q_i(f;\cdot)\|_{\bar{X}_i}$ and $\|P_i(f;\cdot)\|_{\bar{X}_i}$

$$\|R(f;\cdot)-R(f^*;\cdot)\|_{\bar{X}_i} \le \frac{\|P_i(f;\cdot)\|_{\bar{X}_i}\|Q_i(f^*;\cdot)-Q_i(f;\cdot)\|_{\bar{X}_i} + \|Q_i(f;\cdot)\|_{\bar{X}_i}\|P_i(f^*;\cdot)-P_i(f;\cdot)\|_{\bar{X}_i}}{d_i^2/2} \to 0,$$

from which yields what we want to prove.

5. Conclusion

We have presented a few approaches to the construction of generalized rational splines, which seem to be interesting and might find applications in numerical analysis.

The property of allowing confluence, i.e., allowing δ to tend to zero, applies to almost all the constructions offered in this paper. We mention that some of the methods described in Section 2 can easily be extended to the cases in which the vector-valued and matrix-valued rational splines are taken into consideration as treated by Graves-Morris ([4]). Finally we think that the problems related to the singluar structure of rational splines should be studied.

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广义有理样条的几种构造方法

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摘　要

本文利用 Thiele 倒差分方法、Padé 逼近方法、广义 Q.D. 算法及 ε- 算法等构造了几种广义有理样条函数. 此外，通过直接法构造了 $(k-1,k)$- 型广义有理样条，给出了它的行列式表示和余项表示并证明了广义有理样条算子的存在性、唯一性、齐次性及连续性.