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# Concerning Weighted Stirling-type Pairs*

*Leetsch C. Hsu    Yu Hongquan*
(Dalian University of Technology, Dalian 116024)

As is known, a Stirling-type pair may be defined as a pair $A_1(n,k)$, $A_2(n,k)$ of the coefficients in the expansions

$$\frac{1}{k!}(f(t))^k = \sum_{n\geq 0} A_1(n,k)\frac{t^n}{n!}, \quad \frac{1}{k!}(g(t))^k = \sum_{n\geq 0} A_2(n,k)\frac{t^n}{n!}, \tag{1}$$

where $f$ and $g$ are two formal power series over complex field $\mathbf{C}$

$$f(t) = t + \sum_{k\geq 2} a_k t^k, \; g(t) = t + \sum_{k\geq 2} b_k t^k \tag{2}$$

such that $f(g(t)) = g(f(t)) = t$.

Let $W(t)$ be a formal power series with expansions

$$W(t) = 1 + \sum_{k\geq 1} c_k t^k/k!, \quad 1/W(g(t)) = 1 + \sum_{k\geq 1} d_k t^k/k!. \tag{3}$$

Then $\{B_1(n,k), B_2(n,k)\}$ given by the generating functions

$$W(t)\frac{(f(t))^k}{k!} = \sum_{n\geq k} B_1(n,k)\frac{t^n}{n!}, \quad \frac{1}{W(g(t))}\frac{(g(t))^k}{k!} = \sum_{n\geq k} B_2(n,k)\frac{t^n}{n!} \tag{4}$$

is called a weighted Stirling-type pair whenever $f(g(t)) = g(f(t)) = t$.

Accordingly, the Schlömilch-type formula and Lagranges's inverse relation may be written respectively as follows:

$$A_i(n,k) = \sum_{v=0}^{n-k}(-1)^v \binom{2n-k}{n-k-v}\binom{n-1+v}{n-k+v} A_j(n-k+v, v), \tag{5}$$

where $(i,j) = (1,2)$ or $(2,1)$, and

$$\mathbf{C}_{t^n}(g(t))^k == \frac{k}{n}\mathbf{C}_{t^{n-k}}(f(t)/t)^{-n} \tag{6}$$

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Using some analytic and computational techniques, it is not difficult to establish the following results.

**Theorem 1** *For a weighted Stirling-type pair* $\{B_1, B_2\}$, $B_1$ *(or* $B_2$*) can be expressed as a linear combination of* $B_2$*'s (or* $B_1$*'s). Indeed we have*

$$B_1(n,k) = \sum_{j=k}^{n}\sum_{v=0}^{j-k}\sum_{\mu=v}^{j-k+v}(-1)^v\binom{n}{j}\binom{2j-k}{j-k-v}\binom{j-1+v}{j-k+v}\binom{\mu}{v}c_{n-j}c_{\mu-v}B_2(j-k+v,\mu).$$

This will reduce to (5) for the case $W(t)=1$ with $B_i = A_i$ $(i=1,2)$.

**Theorem 2**(Equivalence theorem) *Let* $\{A_1, A_2\}$ *and* $\{B_1, B_2\}$ *be defined by (1) and (4). Then the following statements are equivalent to each other:*

*(i)* $\{B_1, B_2\}$ *is a weighted Stirling-type pair.*
*(ii) The relation* $f(g(t)) = g(f(t)) = t$ *holds.*
*(iii) Inverse relations* $x_n = \sum_k B_1(n,k)y_k \iff y_n = \sum_k B_2(n,k)x_k$ *hold.*
*(iv) Schlömilch-type formula (5) for* $\{A_1, A_2\}$ *holds.*
*(v) Lagrange's relation (6) for* $(f, g)$ *holds.*

**Theorem 3**(Congruence relations) *If* $g(t)$ *and* $W(t)$ *have integer coefficients* $b_k$*'s and* $c_k$*'s as given by (2) and (3), then we have*

$$B_1(p,k) \equiv B_2(p,k) \equiv 0 \pmod p$$

*provided that* $p$ *is an odd prime with* $1 < k < p \le 2k-1$.

For given any four complex parameters $a, b, \alpha, \beta$ with $\alpha \neq \beta$ and $\alpha\beta \neq 0$, a kind of useful weighted Stirling-type pair may be defined by

$$(1+\alpha t)^{(a-b)/\alpha}\left[\frac{(1+\alpha t)^{\beta/\alpha}-1}{\beta}\right]^k = \sum_{n\ge k} S(n,k,a,b;\alpha,\beta)\frac{t^n}{n!}, \tag{7}$$

$$(1+\beta t)^{(b-a)/\beta}\left[\frac{(1+\beta t)^{\alpha/\beta}-1}{\alpha}\right]^k = \sum_{n\ge k} S(n,k,a,b;\alpha,\beta)\frac{t^n}{n!}, \tag{8}$$

**Theorem 4**(Reciprocal relations between polynomial systems) *The pair generated by (7) and (8) satisfies the relations*

$$((t+a)|\alpha)_n = \sum_{k=0}^{n} S(n,k,a,b;\alpha,\beta)((t+b)|\beta)_k, \tag{9}$$

$$((t+b)|\beta)_n = \sum_{k=0}^{n} S(n,k,a,b;\alpha,\beta)((t+a)|\alpha)_k, \tag{10}$$

*where* $((t+a)|\alpha)_0 = 1$, $((t+a)|\alpha)_n = (t+a)(t+a-\alpha)\cdots(t+a-n\alpha+\alpha)$.

**Remark** Note that in (7)–(8) and (9)–(10), $\alpha$ or $\beta$ may be made to tend to zero as limits. As may be observed, various results related to those special weighted Stirling numbers due to Carlitz, Howard and Koutras, et al., respectively, are consequences of the above theorems. Also, it is worth mentioning that Theorems 1 and 2 may be extended to vectorial functions via Lagrange-Good inversion formula and related techniques.