Thus the conditions of Theorem 3 are satisfied. But Theorem 6 of Maiti and Babu[4] is not applicable since f does not satisfy $h(fy, F(f)) \leq h(x, F(f))$ for $x = 1 \overline{\in} X - F(f)$.

Acknowledgements *I am grateful to Professor S.Park for providing me with reprints of his papers which motivated the present work.*

References

[1] J.B.Diaz and F.T.Metcalf, *On the set of subsequential limit points of successive approximations* , Trans. Amer. Math. Soc., **135**(1969), 459–485.

[2] M.Maiti and A.C.Babu, *On subsequential limit points of a sequence of iterates*, Proc. Amer. Math. Soc., **82**(1981), 377–381.

[3] S.Park, *Remarks on subsequential limit points of a sequence of iterates* , J.Korean Math. Soc., **19**(1982), 19–22.

[4] M.Maiti and A.C.Babu, *On subsequential limit points of a sequence of iterates II* , J. Austral. Math. Soc., (Series A) **38**(1985), 118–129.

[5] C.Berge, *Topological spaces* , Oliver & Boyd, London, 1963.

关于迭代序列的子列极限点

刘 泽 庆
(辽宁师范大学数学系，大连 116029)

摘 要

本文讨论了度量空间上一个连续自映射在一点处的迭代序列的子列极限点集的结构，所得的结果统一和推广了 Diaz 和 Metcalf[1], Maiti 和 Babu[2,4] 和 Park[3] 的若干结果.

Journal of Mathematical Research & Exposition
Vol.15, No.4, 519-523, Nov. 1995

On Subsequential Limit Points of A Sequence of Iteraters *

Liu Zeqing
(Dept of Math., Liaoning Normal Univ., Dalian 116029)

Abstract In this paper we show three results concerning the structure of the set of subsequential limit points of a sequence of iterates of a continuous self-map of a metric space. Our results unify and extend corresponding results of Diaz and Metcalf, Maiti and Babu, and Park.

Keywords subsequential limit point, continuous map, metric space, connected set.

Classification AMS(1991) 54H25/CCL 189.1

1. Introduction

Diaz and Metcalf[1] obtained the structure of the set of subsequential limit points of a sequence of iterates of a continuous self-map of a metric space (X,d). Later, Maiti and Babu[2] established a similar result on maps which are contractive over two consecutive elements of an orbit. Park[3] extended a few results of [1,2]. Replacing the distance function by a continuous function on $X\times X$, Maiti and Babu gave a number of variations and generalizations of the results in [1]. Motivated by the results of Park[3] and Maiti and Babu[4], we establish three results on the structure of the set of subsequential limit points of a sequence of iterates of a continuous self-map of a metric space. Our results unify and extend corresponding results of [1–4]. Three illustrative examples are given in support of our results.

Let f be a self-map of a metric space (X,d) and $F(f)$ denote the set of fixed points of f. The orbit of $x\in X$ generated by f is denoted by $O(x,f)$ and its closure by $\bar{O}(x,f)$.$L(x)$ denotes the set of subsequential limit points of the sequence $\{f^n x\}_{n=0}^{\infty}$. Let R_0 be the subspace $[0,\infty)$ of the real line with usual topology, and h a function from $X\times X$ into R_0. For $x\in X$ and $A,B\subset X$, define $d(x,A)=\inf\{d(x,y)|y\in A\}, d(A,B)=\inf\{d(x,y)|x\in A \text{ and } y\in B\}$ and $h(x,A)=\inf\{h(x,y)|y\in A\}$.

2. Lemmas

In order to prove our main theorems, we first show the following Lemmas.

Lemma 1 *Let f be a self-map of a metric space (X,d). Assume that there exists $x\in X$*

*Received Dec.25, 1993. Project Supproted by the Natural Science Foundation of Liaoning Province.

such that $\bar{O}(x,f)$ is compact. Then $d(f^n x, L(x)) \to 0$ as $n \to \infty$.

Proof Suppose that $d(f^n x, L(x)) \not\to 0$ as $n \to \infty$. Then there exists an $\varepsilon > 0$ and a subsequence $\{f^{n_i}x\}_{i=1}^{\infty}$ of $\{f^n x\}_{n=1}^{\infty}$ such that $d(f^{n_i}x, L(x)) \geq \varepsilon$ for $i \geq 1$. The compactness of $\bar{O}(x,f)$ implies that there exists a convergent subsequence $\{f^{k_i}x\}_{i=1}^{\infty}$ of $\{f^{n_i}x\}_{i=1}^{\infty}$. Let $f^{k_i}x \to r$ as $i \to \infty$. Then $r \in L(x)$. Hence $d(f^{k_i}x, L(x)) \to 0$ as $i \to \infty$, which is impossible. Thus $d(f^n x, L(x)) \to 0$ as $n \to \infty$.

Lemma 2 *Let f be a continuous self-map of a metric space (X,d). If there exists $x \in X$ such that $\bar{O}(x,f)$ is compact and $L(x) \subset F(f)$, then*

(1) $d(f^n x, f^{n+1}x) \to 0$ as $n \to \infty$;

(2) $L(x)$ is nonempty, closed and connected.

Proof Suppose that (1) does not hold. Then there exists an $\varepsilon > 0$ and a subsequence $\{f^{n_i}x\}_{i=1}^{\infty}$ of $\{f^n x\}_{n=1}^{\infty}$ such that $d(f^{n_i}x, f^{n_i+1}x) \geq \varepsilon$ for $i \geq 1$. As in the proof of Lemma 1 we can find a subsequence $\{f^{k_i}x\}_{i=1}^{\infty}$ of $\{f^{n_i}x\}_{i=1}^{\infty}$ such that $f^{k_i}x \to r \in L(x)$ as $i \to \infty$. Since f is continuous and $L(x) \subset F(f)$, then $d(f^{n_i}x, f^{n_i+1}x) \to d(r, fr) = 0$ as $i \to \infty$. Consequently $0 \geq \varepsilon > 0$, which is a contradiction. Hence (1) holds.

We now show that (2) holds. The compactness of $\bar{O}(x,f)$ ensures $L(x) \neq \emptyset$. It is easy to see that $L(x)$ is closed. To prove the connectedness of $L(x)$ we assume the contrary, i.e., $L(x) = A \cup B$, where A and B are both nonempty, closed and disjoint. Note that $L(x)$ is a closed subset of the compact set $\bar{O}(x,f)$. Then $L(x)$ is compact. Consequently A and B are also compact and $d(A,B) > 0$. Put $d(A,B) = 3t$. By (1) and Lemma 1 it follows that there exists a positive integer N such that $\max\{d(f^n x, f^{n+1}x), d(f^n x, L(x))\} < t$ for $n \geq N$. Since $L(x) = A \cup B$ is compact, there exists $w \in A \cup B$ such that $d(f^n x, A \cup B) = d(f^n x, w)$. If $w \in A$, then $d(f^n x, A) \leq d(f^n x, w) < t$. Consequently, for any $n \geq N$, either $d(f^n x, A) < t$ or $d(f^n x, B) < t$. But both these inequalities cannot hold simultaneously for the same n because in that case

$$0 < 3t = d(A,B) \leq d(f^n x, A) + d(f^n x, B) < 2t$$

which is impossible. The set of positive integers $n \geq N$, such that $d(f^n x, A) < t$, is not empty, because $\emptyset \neq A \subset L(x)$. Similarly, the set of positive integers $n \geq N$, such that $d(f^n x, B) < t$, is also not empty. Hence there exists a positive integer $n \geq N$ such that both $d(f^n x, A) < t$ and $d(f^{n+1}x, B) < t$. Consequently,

$$3t = d(A,B) \leq d(A, f^n x) + d(f^n x, f^{n+1}x) + d(f^{n+1}x, B) < 3t$$

which is a contradiction. Hence $L(x)$ is connected. Thus (2) holds.

Lemma 3 *Let f be a self-map of a metric space (X,d), and $F(f)$ nonempty and compact. Assume that there exists a continuous function $h : X \times X \to R_0$ such that $h(x,y) = 0$ implies $x = y$. If there exists $x \in X$ such that $h(f^n x, F(f)) \to 0$ as $n \to \infty$, then $L(x) \subset F(f)$.*

Proof Let $p \in L(x)$. Then there exists a subsequence $\{f^{n_i}x\}_{i=1}^{\infty}$ of $\{f^n x\}_{n=1}^{\infty}$ such that $f^{n_i}x \to p$ as $i \to \infty$. By the continuity of h and compactness of $F(f)$ we can find

$a_{n_i} \in F(f)$ such that $h(f^{n_i}x, F(f) = h(f^{n_i}x, a_{n_i})$ for each $i \geq 1$. Since $F(f)$ is compact, there exists a subsequence $\{a_{k_i}\}_{i=1}^{\infty}$ of $\{a_{n_i}\}_{i=1}^{\infty}$ such that $a_{k_i} \to a \in F(f)$ as $i \to \infty$. Consequently, we obtain

$$h(f^{k_i}x, F(f)) = h(f^{k_i}x, a_{k_i}) \to h(p, a) = 0 \text{ as } i \to \infty$$

which implies that $p = a \in F(f)$; i.e., $L(x) \subset F(f)$.

3. Results and Examples

Our main results are as follows.

Theorem 1 *Let f be a continuous self-map of a metric space $(X, d), h$ a continuous function from $X \times X$ into R_0 such that $h(x, y) = 0$ implies $x = y$. Assume that $h(f^n x, f^{n+1}x) \to 0$ as $n \to \infty$ and $\bar{O}(x, f)$ is compact for some $x \in X$. Then $L(x)$ is a nonempty, closed and connected subset of $F(f)$, and either*

(a) $L(x)$ is a singleton, and $\lim_{n\to\infty} f^n x$ exists and belonges to $F(f)$, or

(b) $L(x)$ is uncountable, and it is contained in the boundary of $F(f)$.

Proof Let $p \in L(x)$. Then there exists a subsequence $\{f^{n_i}x\}_{i=1}^{\infty}$ of $\{f^n x\}_{n=1}^{\infty}$ such that $f^{n_i}x \to p$ as $i \to \infty$. Hence $h(f^{n_i}x, f^{n_i+1}x) \to h(p, fp) = 0$ as $i \to \infty$, because f and h are continuous. This implies that $p = fp \in F(f)$; i.e., $L(x) \subset F(f)$. It follows from Lemma 2 that $L(x)$ is a nonempty, closed and connected subset of $F(f)$. Note that $\bar{O}(x, f)$ is compact and $L(x) \subset \bar{O}(x, f)$. Hence $L(x)$ is also compact. By Theorem 1 of Berge [5,p.96] it follows that $L(x)$ is either a singleton or uncountable. Clearly (a) follows from Lemma 1. The rest of the proof is exactly the same as that of Theorem 2 of Diaz and Metcalf[1]

Remark 1 For $h = d$, Theorem 1 is due to Park [3,Theorem 1]. In case $F(f)$ is compact, Theorem 1 is due to Maiti and Babu [4,Theorem 3]. It is easy to see that Theorem 3 of Maiti and Babu[2] and Theorem 6 of Diaz and Metcalf[1] are special cases of Theorem 1. The following examples reveal that Theorem 1 extends properly Theorem 3 of Maiti and Babu[2], Theorem 6 of Diaz and Metcalf[1] and Theorem 3 of Maiti and Babu[4].

Example 1 Let $X = (-\infty, 0] \cup \{\frac{1}{n} | n \geq 1\}$ with the usual metric. Define $f : X \to X$ by $fx = x$ for $x \in (-\infty, 0]$ and $f\frac{1}{n} = \frac{1}{n+1}$ for $n \geq 1$. Note that $F(f) = (-\infty, 0]$ and $fX = (-\infty, 0] \cup \{\frac{1}{n} | n \geq 2\}$. Consequently $F(f)$ and fX are not compact. Hence Theorem 6 of Diaz and Metcalf[1] and Theorem 3 of Maiti and Babu[4] are not applicable. Let $h(x, y) = (x - y)^2(|x| + 1)$ for $(x, y) \in X \times X$. Then

$$h(f^n 1, f^{n+1}1) = h(\frac{1}{n+1}, \frac{1}{n+2}) = (\frac{1}{n+1} - \frac{1}{n+2})^2(\frac{1}{n+1} + 1) \to 0 \text{ as } n \to \infty$$

and $\bar{O}(1, f) = \{0\} \cup \{\frac{1}{n} | n \geq 1\}$ is compact. Obviously the conditions of Theorem 1 are satisfied.

Example 2 Let $X = \{1, 2, 3\}$ with the metric $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. Define $f : X \to X$ by $f1 = 2$ and $f2 = f3 = 3$. Then f is continuous and $\bar{O}(1, f) = X$ is compact. It is easy to check that the following conditions

(i) $d(x, fx) + d(y, fy) < 2d(x, y)$,

(ii) $d(x, fx) + d(y, fy) < \frac{2}{3}\{d(x, fy) + d(y, fx) + d(x, y)\}$,

(iii) $d(x, fx) + d(y, fy) + d(fx, fy) < \frac{2}{3}\{d(x, fy) + d(y, fx) + d(x, y)\}$,

(iv) $d(fx, fy) < \max\{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy) + d(y, fx)]\}$

do not hold for $x = 1$ and $y = 2$. Thus Theorem 3 of Maiti and Babu[2] is not applicable. Let $h = d$. Then $d(f^n 1, f^{n+1} 1) \to 0$ as $n \to \infty$. Hence the assumptions of Theorem 1 are satisfied.

Theorem 2 *Let f be a continuous self-map of a metric space (X, d), h a continuous function from $X \times X$ into R_0 such that $h(x, y) = 0$ implies $x = y$. Assume that $\bar{O}(x, f)$ is compact for some $x \in X$. If f satisfies $h(fy, f^2 y) < h(y, fy)$ for each $y \in \bar{O}(x, f)$ and $y \neq fy$, then the conclusion of Theorem 1 holds.*

Proof Setting $a_n = h(f^n x, f^{n+1} x)$, we have $a_{n+1} \le a_n$. Hence $a_n \to r = \inf\{a_n | n \ge 1\}$ as $n \to \infty$. Let $p \in L(x)$. Then there exists a subsequence $\{f^{n_i} x\}_{i=1}^{\infty}$ converges to p. By the continuity of f we have $f^{n_i+1} x \to fp$ and $f^{n_i+2} x \to f^2 p$ as $i \to \infty$. Hence

$$r = \lim_{i \to \infty} h(f^{n_i} x, f^{n_i+1} x) = h(p, fp) = h(fp, f^2 p) = \lim_{i \to \infty} h(f^{n_i+1} x, f^{n_i+2} x).$$

Suppose that $p \neq fp$. By the assumption we have $h(fp, f^2 p) < h(p, fp)$, which is absurd. Hence $p = fp$ and $a_n \to r = 0$ as $n \to \infty$. Thus Theorem 2 follows from Theorem 1.

Remark 2 In case $h = d$, Theorem 2 is reduced to Theorem 2 of Park[3].

Theorem 3 *Let f be a continuous self-map of a metric space (X, d), and $F(f)$ nonempty and compact. Assume that there exists a continuous function $h : X \times X \to R_0$ such that $h(x, y) = 0$ implies $x = y$. If for some $x \in X, \bar{O}(x, f)$ is compact and $h(f^n x, F(f)) \to 0$ as $n \to \infty$, then $L(x)$ is a nonempty, compact and connected subset of $F(f)$, and the rest of the conclusion of Theorem 1 remains unchanged.*

Proof Note that $F(f)$ is compact. By Lemmas 2 and 3 it follows that $L(x)$ is a nonempty, compact and connected subset of $F(f)$. The remaining part of the proof is as in Theorem 1.

Remark 3 It is easy to see that Theorem 6 of Maiti and Babu[4] follows from Theorem 3. The following example shows that Theorem 3 is a proper generalization of Theorem 6 of Maiti and Babu[4].

Example 3 Let $X = \{0, 3\} \cup \{\frac{1}{n} | n \ge 1\}$ with the usual metric. Define $f : X \to X$ by $f0 = 0, f1 = 3, f3 = \frac{1}{2}$ and $f\frac{1}{n} = \frac{1}{n+1}$ for $n \ge 2$. Let $h(x, y) = |x - y|(x + y + 1)$ for $x, y \in X \times X$. Then $F(f) = \{0\}$ and $\bar{O}(1, f) = X$. Clearly $F(f)$ and $\bar{O}(1, f)$ are compact, and f and h are continuous. For $n \ge 2$, we have

$$h(f^n 1, F(f)) = h(\frac{1}{n}, 0) = \frac{n+1}{n^2} \to \text{ as } n \to \infty.$$

Thus the conditions of Theorem 3 are satisfied. But Theorem 6 of Maiti and Babu[4] is not applicable since f does not satisfy $h(fy, F(f)) \leq h(x, F(f))$ for $x = 1 \overline{\in} X - F(f)$.

Acknowledgements *I am grateful to Professor S.Park for providing me with reprints of his papers which motivated the present work.*

References

[1] J.B.Diaz and F.T.Metcalf, *On the set of subsequential limit points of successive approximations* , Trans. Amer. Math. Soc., **135**(1969), 459–485.

[2] M.Maiti and A.C.Babu, *On subsequential limit points of a sequence of iterates*, Proc. Amer. Math. Soc., **82**(1981), 377–381.

[3] S.Park, *Remarks on subsequential limit points of a sequence of iterates* , J.Korean Math. Soc., **19**(1982), 19–22.

[4] M.Maiti and A.C.Babu, *On subsequential limit points of a sequence of iterates II* , J. Austral. Math. Soc., (Series A) **38**(1985), 118–129.

[5] C.Berge, *Topological spaces* , Oliver & Boyd, London, 1963.

关于迭代序列的子列极限点

刘 泽 庆
(辽宁师范大学数学系，大连 116029)

摘　要

本文讨论了度量空间上一个连续自映射在一点处的迭代序列的子列极限点集的结构，所得的结果统一和推广了 Diaz 和 Metcalf[1], Maiti 和 Babu[2,4] 和 Park[3] 的若干结果.