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关于两参数Wiener过程增量的一些结果*

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摘 要 本文讨论了两参数 Wiener 过程增量的一些结果. 相应于[1]的讨论，可找出正则化因子 μ_T，使得 $\inf\limits_{0\le x\le T-a_T}\inf\limits_{0\le y\le T-b_T}\sup\limits_{0\le s\le a_T}\sup\limits_{0\le t\le b_T}|W([x,x+s]\times[y,y+t])|$ 的上极限为 1. 进一步，又给出了较一般的增量的上极限以及它的滞后增量的上极限.

关键词 两参数 Wiener 过程，正则化因子，增量，滞后增量.

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一 引 言

[1]中讨论了两参数 Wiener 过程增量有多小的问题，并得到如下定理：

定理A 设 $0<a_T,b_T\le T$，且 $a_T,b_T,\frac{T}{a_T}$ 和 $\frac{T}{b_T}$ 都是 T 的不减函数，$R=[x,x+s]\times[y,y+t]$，则对于两参数 Wiener 过程 $\{W(s,t);s,t\ge 0\}$，有

$$\varliminf_{T\to\infty}\inf_{0\le x\le T-a_T}\inf_{0\le y\le T-b_T}\sup_{0\le s\le a_T}\sup_{0\le t\le b_T}\mu_T|W(R)|=1\quad \text{a.s.}\tag{1}$$

其中 $\mu_T=\left(\dfrac{8(\log\dfrac{T^2}{a_Tb_T}+\log\log T)}{\pi^2a_Tb_T}\right)^{1/2}$.

进一步，若还有 $\lim\limits_{T\to\infty}\dfrac{\log\dfrac{T^2}{a_Tb_T}}{\log\log T}=\infty$，则(1)中 $\varliminf$ 可换成 lim.

定理B 设 $0<a_T\le T$，a_T 和 $\frac{T}{a_T}$ 是 T 的不减函数，$R=[x_1,x_2]\times[y_1,y_2]$，则

$$\varliminf_{T\to\infty}\inf_{0\le t\le T-a_T}\sup_{R\in L_t(a_T)}\gamma_T|W(R)|=1\quad \text{a.s.}\tag{2}$$

其中 $\gamma_T=\left(\dfrac{8(\log\dfrac{T}{a_T}+\log\log T)}{\pi^2a_T}\right)^{1/2}$，而

$$L_t(a_T)=\{R:0\le x_i,y_i\le\sqrt{T},x_1y_1=t,\lambda(R)=(x_2-x_1)(y_2-y_1)\le a_T\}.$$

定理C 设 $0<a_T\le T$，有

$$\varliminf_{T\to\infty}\inf_{a_T\le t\le T}\inf_{t\le s\le T}\sup_{R\in M_s(t)}\rho(T,t)|W(R)|=1\quad \text{a.s.}\tag{3}$$

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其中 $\rho(T,t)=\left(\frac{8(\log\frac{T}{t}+\log\log t)}{\pi^2 t}\right)^{1/2}$,而

$$M_s(t)=\{R:0\leqslant x_i,y_i\leqslant\sqrt{s},x_2y_2=s,\lambda(R)\leqslant t\}$$

定理 A,B,C 都是讨论这三种增量的下极限,很自然的一个问题是这三种增量的上极限是什么? 本文回答了这个问题.

二 结论与证明

定理 1 在定理 A 的条件下若还有

$$\lim_{T\to\infty}\frac{\log\frac{T^2}{a_Tb_T}}{\log\log T}=r,\quad 0<r<\infty, \tag{4}$$

则有

$$\varlimsup_{T\to\infty}\inf_{0\leqslant x\leqslant T-a_T}\inf_{0\leqslant y\leqslant T-b_T}\sup_{0\leqslant s\leqslant a_T}\sup_{0\leqslant t\leqslant b_T}v_T|W(R)|=1\quad \text{a.s.} \tag{5}$$

其中 $v_T=\left(\frac{8\log\frac{T^2}{a_Tb_T}}{\pi^2 a_Tb_T}\right)^{1/2}$.

证明 第一步证明

$$\varlimsup_{T\to\infty}\inf_{0\leqslant x\leqslant T-a_T}\inf_{0\leqslant y\leqslant T-b_T}\sup_{0\leqslant s\leqslant a_T}\sup_{0\leqslant t\leqslant b_T}v_T|W(R)|\leqslant 1\quad \text{a.s.} \tag{6}$$

为此我们取 T 的一个子列 $\{T_n,n=1,2,\cdots\}$,使得 $T_n=\theta^n,\theta>1,\theta$ 待定. 由 $\frac{T_{n+1}^2a_{T_n}b_{T_n}}{a_{T_{n+1}}b_{T_{n+1}}T_n^2}\geqslant 1$ 有

$$1\leqslant\frac{a_{T_{n+1}}b_{T_{n+1}}}{a_{T_n}b_{T_n}}\leqslant\frac{T_{n+1}^2}{T_n^2}=\theta^2.$$

并且当 θ 充分小,n 充分大时,对$\forall\ \varepsilon>0$,有

$$\frac{1}{1+\varepsilon}\frac{a_{T_{n+1}}b_{T_{n+1}}}{a_{T_n}b_{T_n}}\leqslant 1-\varepsilon',\quad \varepsilon'>0.$$

对固定的 $n\geqslant 1$,令

$$x_i=ia_{T_{n+1}},\ i=0,1,\cdots,p_n,p_n=\left[\frac{T_n}{a_{T_{n+1}}}\right]-1$$

$$y_i=jb_{T_{n+1}},\ j=0,1,\cdots,Q_n,Q_n=\left[\frac{T_n}{b_{T_{n+1}}}\right]-1$$

记 $R_{ij}=[x_i,x_i+s]\times[y_j,y_j+t]$,于是由[2]中引理有

$$\sum_{n=1}^{\infty}P\{A_n>1+\varepsilon\}$$

$$\triangleq\sum_{n=1}^{\infty}P\left\{\inf_{0\leqslant x\leqslant T_n-a_{T_{n+1}}}\inf_{0\leqslant y\leqslant T_n-b_{T_{n+1}}}\sup_{0\leqslant s\leqslant a_{T_{n+1}}}\sup_{0\leqslant t\leqslant b_{T_{n+1}}}\left(\frac{8\log\frac{T_{n+1}^2}{a_{T_{n+1}}b_{T_{n+1}}}}{\pi^2a_{T_n}b_{T_n}}\right)^{1/2}|W(R)|>1+\varepsilon\right\}$$

$$\leqslant \sum_{n=1}^{\infty} P\left\{\min_{0\leqslant i\leqslant P_n}\min_{0\leqslant j\leqslant Q_n}\sup_{0\leqslant s\leqslant a_{T_{n+1}}}\sup_{0\leqslant t\leqslant b_{T_{n+1}}}\left(\frac{8\log\frac{T_{n+1}^2}{a_{T_{n+1}}b_{T_{n+1}}}}{\pi^2 a_{T_n}b_{T_n}}\right)^{1/2}|W(R_{ij})| > 1+\varepsilon\right\}$$

$$\leqslant \sum_{n=1}^{\infty}\exp\{-cn^{\delta}\} < \infty, \quad \delta > 0.$$

从而

$$\overline{\lim_{n\to\infty}}\inf_{0\leqslant x\leqslant T_n-a_{T_{n+1}}}\inf_{0\leqslant y\leqslant T_n-b_{T_{n+1}}}\sup_{0\leqslant s\leqslant a_{T_{n+1}}}\sup_{0\leqslant t\leqslant b_{T_{n+1}}}\left(\frac{8\log\frac{T_{n+1}^2}{a_{T_{n+1}}b_{T_{n+1}}}}{\pi^2 a_{T_n}b_{T_n}}\right)^{1/2}|W(R)| \leqslant 1.$$

由于 $T-a_T\geqslant T_n-a_{T_{n+1}}, T-b_T\geqslant T_n-b_{T_{n+1}}, a_{T_n}\leqslant a_T\leqslant a_{T_{n+1}}, b_{T_n}\leqslant b_T\leqslant b_{T_{n+1}}$，我们可推得(6)成立，即第一步证毕.

第二步证明：

$$\overline{\lim_{T\to\infty}}\inf_{0\leqslant x\leqslant T-a_T}\inf_{0\leqslant y\leqslant T-b_T}\sup_{0\leqslant s\leqslant a_T}\sup_{0\leqslant t\leqslant b_T} v_T|W(R)| \geqslant 1 \quad \text{a.s.} \tag{7}$$

为此令 $T_n=e^{n^2}, n=1,2,\cdots$，对固定的 n 令

$$x_i = ia_{T_n}(\varphi(T_n))^{-3}, i=0,1,\cdots,p_n, \quad p_n=\left[\frac{T_n}{a_{T_n}}(\varphi(T_n))^3\right]-1,$$

$$y_j = jb_{T_n}(\varphi(T_n))^{-3}, j=0,1,\cdots,q_n, \quad q_n=\left[\frac{T_n}{b_{T_n}}(\varphi(T_n))^3\right]-1,$$

这里 $\varphi(T)=\log\frac{T^2}{a_Tb_T}+\log\log T$. 于是由[2]中引理有

$$\sum_{n=1}^{\infty} P\left\{\min_{0\leqslant i\leqslant p_n}\min_{0\leqslant j\leqslant q_n}\sup_{0\leqslant s\leqslant a_{T_n}}\sup_{0\leqslant t\leqslant b_{T_n}}\left(\frac{8\log\frac{T_n^2}{a_{T_n}b_{T_n}}}{\pi^2 a_{T_n}b_{T_n}}\right)^{1/2}|W(R_{ij})| \leqslant \sqrt{1-\varepsilon}\right\}$$

$$\leqslant c\sum_{n=1}^{\infty}\frac{1}{n^{1+\delta}} < \infty, \quad \delta > 0,$$

所以有

$$\underline{\lim_{n\to\infty}}\min_{0\leqslant i\leqslant p_n}\min_{0\leqslant j\leqslant q_n}\sup_{0\leqslant s\leqslant a_{T_n}}\sup_{0\leqslant t\leqslant b_{T_n}}\left(\frac{8\log\frac{T_n^2}{a_{T_n}b_{T_n}}}{\pi^2 a_{T_n}b_{T_n}}\right)^{1/2}|W(R_{ij})| \geqslant 1 \quad \text{a.s.} \tag{8}$$

记 $R^{(k)}$ 为[1]中(15)式的各矩形，于是由[1]中(16)式有

$$\overline{\lim_{n\to\infty}}\max_{\substack{0\leqslant i\leqslant p_n\\ 0\leqslant j\leqslant q_n}}\sup_{\substack{x_i\leqslant x\leqslant x_{i+1}\\ y_j\leqslant y\leqslant y_{j+1}}}\sup_{\substack{0\leqslant s\leqslant a_{T_n}\\ 0\leqslant t\leqslant b_{T_n}}}\left(\frac{8\log\frac{T_n^2}{a_{T_n}b_{T_n}}}{\pi^2 a_{T_n}b_{T_n}}\right)^{1/2}|W(R^{(k)})| = 0 \quad \text{a.s.} \tag{9}$$

进而有

$$\underline{\lim_{n\to\infty}}\inf_{0\leqslant x\leqslant T_n-a_{T_n}}\inf_{0\leqslant y\leqslant T_n-b_{T_n}}\sup_{0\leqslant s\leqslant a_{T_n}}\sup_{0\leqslant t\leqslant b_{T_n}}\left(\frac{8\log\frac{T_n^2}{a_{T_n}b_{T_n}}}{\pi^2 a_{T_n}b_{T_n}}\right)^{1/2}|W(R)|$$

$$\geqslant \lim_{n\to\infty}\min_{0\leqslant i\leqslant p_n}\min_{0\leqslant j\leqslant q_n}\sup_{0\leqslant s\leqslant a_{T_n}}\sup_{0\leqslant t\leqslant b_{T_n}}\left(\frac{8\log\dfrac{T_n^2}{a_{T_n}b_{T_n}}}{\pi^2 a_{T_n}b_{T_n}}\right)^{1/2}|W(R_{ij})|$$

$$-\sum_{k=1}^{6}\overline{\lim_{n\to\infty}}\max_{\substack{0\leqslant i\leqslant p_n\\ 0\leqslant j\leqslant q_n}}\sup_{\substack{x_i\leqslant x\leqslant x_{i+1}\\ y_j\leqslant y\leqslant y_{j+1}}}\sup_{\substack{0\leqslant s\leqslant a_{T_n}\\ 0\leqslant t\leqslant b_{T_n}}}\left(\frac{8\log\dfrac{T_n^2}{a_{T_n}b_{T_n}}}{\pi^2 a_{T_n}b_{T_n}}\right)^{1/2}|W(R^{(k)})|\geqslant 1. \tag{10}$$

因而更有

$$\overline{\lim_{n\to\infty}}\inf_{0\leqslant x\leqslant T_n-a_{T_n}}\inf_{0\leqslant y\leqslant T_n-b_{T_n}}\sup_{0\leqslant s\leqslant a_{T_n}}\sup_{0\leqslant t\leqslant b_{T_n}}\left(\frac{8\log\dfrac{T_n^2}{a_{T_n}b_{T_n}}}{\pi^2 a_{T_n}b_{T_n}}\right)^{1/2}|W(R)|\geqslant 1, \tag{11}$$

于是(7)式成立,第二步证毕.

系 在定理 1 的条件下,有

$$\overline{\lim_{T\to\infty}}\inf_{0\leqslant x\leqslant T-a_T}\inf_{0\leqslant y\leqslant T-b_T}\sup_{0\leqslant s\leqslant a_T}\sup_{0\leqslant t\leqslant b_T}\mu_T|W(R)| = \left(\frac{r+1}{r}\right)^{1/2}. \tag{12}$$

定理 2 在定理 B 的条件下,若

i) $\lim\limits_{T\to\infty}\dfrac{\log\dfrac{T}{a_T}}{\log\log T}=r, 0<r<\infty$,则

$$\overline{\lim_{n\to\infty}}\inf_{0\leqslant t\leqslant T-a_T}\sup_{R\in L_t(a_T)}\left(\frac{8\log T/a_T}{\pi^2 a_T}\right)^{1/2}|W(R)| = 1 \quad \text{a.s.} \tag{13}$$

ii) $\lim\limits_{T\to\infty}\dfrac{\log\dfrac{T}{a_T}}{\log\log T}=\infty$,则

$$\lim_{T\to\infty}\inf_{0\leqslant t\leqslant T-a_T}\sup_{R\in L_t(a_T)}\left(\frac{8\log T/a_T}{\pi^2 a_T}\right)^{1/2}|W(R)| = 1 \quad \text{a.s.} \tag{14}$$

证明 i) 证明(13)成立.

第一步证明:

$$\overline{\lim_{T\to\infty}}\inf_{0\leqslant t\leqslant T-a_T}\sup_{R\in L_t(a_T)}\left(\frac{8\log T/a_T}{\pi^2 a_T}\right)^{1/2}|W(R)| \leqslant 1 \quad \text{a.s.} \tag{15}$$

为此取 $T_n=\theta^n, \theta>1, \theta$ 待定. 记 $R=[x, x+u]\times[y, y+u]$,对固定的 n,用 $\dfrac{a_{T_n}}{\sqrt{T_n}}(\varphi(T_n))^{-3}$ 来分割区间 $[0,\sqrt{T_n}]$,各分点为

$$0, \frac{a_{T_n}}{\sqrt{T_n}}(\varphi(T_n))^{-3}, 2\frac{a_{T_n}}{\sqrt{T_n}}(\varphi(T_n))^{-3}, \cdots, J_n\frac{a_{T_n}}{\sqrt{T_n}}(\varphi(T_n))^{-3}$$

这里 $\varphi(T_n)=\log\dfrac{T_n}{a_{T_n}}+\log\log T_n, J_n=\left[\dfrac{T_n}{a_{T_n}}(\varphi(T_n))^3\right]+1$.

现在考察以下概率:

$$P(A_n)\overset{\triangle}{=}P\{\max_{1\leqslant j\leqslant J_{n+1}}\inf_{0\leqslant t\leqslant T_n-a_{T_{n+1}}}\sup_{0\leqslant u\leqslant j\frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3}}\sup_{0\leqslant v\leqslant\frac{\sqrt{T_{n+1}}}{j(\varphi(T_n))^{-3}}}\left(\frac{8\log\frac{T_{n+1}}{a_{T_{n+1}}}}{\pi^2a_{T_n}}\right)^{1/2}|W(R)|\geqslant\sqrt{1+\varepsilon}\}$$

$$\leqslant\sum_{j=1}^{J_{n+1}}P\{\inf_{0\leqslant x\leqslant\sqrt{T_n}-j\frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3}}\inf_{0\leqslant y\leqslant\sqrt{T_n}-\frac{\sqrt{T_{n+1}}}{j(\varphi(T_n))^{-3}}}\sup_{0\leqslant u\leqslant j\frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3}}$$

$$\cdot\sup_{0\leqslant v\leqslant\frac{\sqrt{T_{n+1}}}{j(\varphi(T_n))^{-3}}}\left(\frac{8\log\frac{T_{n+1}}{a_{T_{n+1}}}}{\pi^2a_{T_n}}\right)^{1/2}|W(R)|>1+\varepsilon\}\overset{\triangle}{=}\sum_{j=1}^{J_{n+1}}P(A_n^{(j)}).$$

对于 $P(A_n^{(j)})$，令

$$x_i=i\left(j\frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3}\right),i=0,1,\cdots,p_n^{(j)},\quad p_n^{(j)}=\left[\frac{\sqrt{T_nT_{n+1}}}{ja_{T_{n+1}}}(\varphi(T_n))^3\right]-1,$$

$$y_k=k\left(\frac{\sqrt{T_{n+1}}}{j(\varphi(T_n))^{-3}}\right),k=0,1,\cdots,Q_n^{(j)},\quad Q_n^{(j)}=\left[j\sqrt{\frac{T_n}{T_{n+1}}}(\varphi(T_n))^{-3}\right]-1.$$

记 $R_{ik}=[x_i,x_i+u]\times[y_k,y_k+v]$，于是由[2]中引理有

$$P(A_n^{(j)})\leqslant P\{\min_{0\leqslant i\leqslant p_n^{(j)}}\min_{0\leqslant k\leqslant Q_n^{(j)}}\sup_{0\leqslant u\leqslant j\frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3}}\sup_{0\leqslant v\leqslant\frac{\sqrt{T_{n+1}}}{j(\varphi(T_n))^{-3}}}\left(\frac{8\log\frac{T_{n+1}}{a_{T_{n+1}}}}{\pi^2a_{T_n}}\right)^{1/2}|W(R_{ik})|>\sqrt{1+\varepsilon}\}$$

$$\leqslant\exp\{-\frac{2}{\pi}\left(\frac{a_{T_{n+1}}}{T_{n+1}}\right)^{\frac{1}{1+\varepsilon}\frac{a_{T_{n+1}}}{a_{T_n}}}\frac{T_n}{a_{T_{n+1}}}\}.$$

由 $\dfrac{\frac{T_{n+1}}{a_{T_{n+1}}}}{\frac{T_n}{a_{T_n}}}\geqslant1$ 知 $\dfrac{a_{T_{n+1}}}{a_{T_n}}\leqslant\dfrac{T_{n+1}}{T_n}=\theta$，因而当 θ 充分接近于 1 时，存在 $\varepsilon'>0$，使得 $\dfrac{1}{1+\varepsilon}\dfrac{a_{T_{n+1}}}{a_{T_n}}\leqslant1-\varepsilon'$.

故

$$\sum_{n=1}^{\infty}P(A_n)\leqslant\sum_{n=1}^{\infty}\frac{T_{n+1}}{a_{T_{n+1}}}(\varphi(T_{n+1}))^3\exp\{-\frac{2}{\pi}\left(\frac{a_{T_{n+1}}}{T_{n+1}}\right)^{\frac{1}{1+\varepsilon}\frac{a_{T_{n+1}}}{a_{T_n}}}\frac{T_n}{a_{T_{n+1}}}\}$$

$$\leqslant\sum_{n=1}^{\infty}\frac{T_{n+1}}{a_{T_{n+1}}}(\varphi(T_{n+1}))^3\exp\{-c\left(\frac{T_{n+1}}{a_{T_{n+1}}}\right)^{\varepsilon'}\}\leqslant\sum_{n=1}^{\infty}c_1\exp\{-c_2(n+1)^{\delta}\}<\infty,\quad\delta>0.$$

从而

$$\varlimsup_{n\to\infty}\max_{1\leqslant j\leqslant J_{n+1}}\inf_{0\leqslant t\leqslant T_n-a_{T_{n+1}}}\sup_{0\leqslant u\leqslant j\frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3}}\sup_{0\leqslant v\leqslant\frac{\sqrt{T_{n+1}}}{j(\varphi(T_n))^{-3}}}\left(\frac{8\log\frac{T_{n+1}}{a_{T_{n+1}}}}{\pi^2a_{T_n}}\right)^{1/2}|W(R)|\leqslant1.\qquad(16)$$

若 $R=[x,x+u]\times[y,y+v]$，则对任一 $R\in L_t(a_{T_{n+1}})$，都有 $xy=t,uv\leqslant a_{T_{n+1}},0\leqslant x,y\leqslant$

$\sqrt{T_{n+1}}, 0 \leqslant x+u, y+v \leqslant \sqrt{T_{n+1}}$,从而存在一个 $j, 0 \leqslant j \leqslant T_{n+1}$,使得

$$j \frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3} \leqslant u \leqslant (j+1) \frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3}.$$

此时 R 可分为两部分,一部分 R_1 包含在矩形

$$\left[x, x+j \frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3}\right] \times \left[y, y+\frac{\sqrt{T_{n+1}}}{j(\varphi(T_n))^{-3}}\right]$$

之中,另一部分 R_2 包含在矩形

$$\left[x+j \frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3}, x+(j+1) \frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3}\right] \times \left[y, y+\frac{\sqrt{T_{n+1}}}{j(\varphi(T_n))^{-3}}\right]$$

之中. 于是有

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \inf_{0\leqslant t\leqslant T_n-a_{T_{n+1}}} \sup_{R\in L_t(a_{T_{n+1}})} \left(\frac{8\log \frac{T_{n+1}}{a_{T_{n+1}}}}{\pi^2 a_{T_n}}\right)^{1/2} |W(R)| \\
\leqslant{}& \overline{\lim_{n\to\infty}} \max_{1\leqslant j\leqslant J_{n+1}} \inf_{0\leqslant t\leqslant T_n-a_{T_{n+1}}} \sup_{0\leqslant u_1\leqslant j\frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3}} \sup_{0\leqslant v_1\leqslant \frac{\sqrt{T_{n+1}}}{j(\varphi(T_n))^{-3}}} \left(\frac{8\log \frac{T_{n+1}}{a_{T_{n+1}}}}{\pi^2 a_{T_n}}\right)^{1/2} |W(R_1)| \\
&+\overline{\lim_{n\to\infty}} \sup_{0\leqslant x\leqslant\sqrt{T_{n+1}}} \sup_{0\leqslant y\leqslant\sqrt{T_{n+1}}} \sup_{0\leqslant u_2\leqslant \frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3}} \sup_{\substack{0\leqslant v_2\leqslant\sqrt{T_{n+1}}\\ 0\leqslant y+v_2\leqslant\sqrt{T_{n+1}}}} \left(\frac{8\log \frac{T_{n+1}}{a_{T_{n+1}}}}{\pi^2 a_{T_n}}\right)^{1/2} |W(R_2)| \stackrel{\triangle}{=} I_1+I_2.
\end{aligned} \tag{17}$$

由(16)知 $I_1 \leqslant 1$,对于 I_2 利用[4]定理 1.12.5 就有

$$\begin{aligned}
I_2 = {}& \overline{\lim_{n\to\infty}} \sup_{0\leqslant x\leqslant\sqrt{T_{n+1}}} \sup_{0\leqslant y\leqslant\sqrt{T_{n+1}}} \sup_{\substack{0\leqslant u_2\leqslant\sqrt{T_{n+1}}\\ 0\leqslant x+u_2\leqslant\sqrt{T_{n+1}}}} \sup_{\substack{0\leqslant v_2\leqslant\sqrt{T_{n+1}}\\ 0\leqslant y+v_2\leqslant\sqrt{T_{n+1}}}} \beta_{T_n} |W(R)| \\
&\times \lim_{n\to\infty} \left(\frac{8\log \frac{T_{n+1}}{a_{T_{n+1}}}}{\pi^2 a_{T_n}}\right)^{1/2} \cdot \beta_{T_n}^{-1} = 0,
\end{aligned} \tag{18}$$

这里 $\beta_{T_n} = \left(2 \frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3}\sqrt{T_{n+1}}\left(\log \frac{T_{n+1}}{a_{T_{n+1}}(\varphi(T_n))^{-3}} + \log\log \sqrt{T_{n+1}}\right)\right)^{-\frac{1}{2}}$. 由(16)、(17)、(18)可推得(15)成立.

第二步证明

$$\overline{\lim_{T\to\infty}} \inf_{0\leqslant t\leqslant T_n-a_T} \sup_{R\in L_t(a_T)} \left(\frac{8\log \frac{T}{a_T}}{\pi^2 a_T}\right)^{1/2} |W(R)| \geqslant 1 \quad \text{a.s.} \tag{19}$$

利用定理 1,有

$$\overline{\lim_{T\to\infty}}\inf_{0\leqslant t\leqslant T_n-a_T}\sup_{R\in L_t(a_T)}\left(\frac{8\log\frac{T}{a_T}}{\pi^2 a_T}\right)^{1/2}|W(R)|\geqslant\overline{\lim_{T\to\infty}}\inf_{\substack{0\leqslant x\leqslant\sqrt{T}\\0\leqslant y\leqslant\sqrt{T}}}\sup_{\substack{0\leqslant s\leqslant\sqrt{a_T}\\0\leqslant t\leqslant\sqrt{a_T}}}\left(\frac{8\log\frac{T}{a_T}}{\pi^2 a_T}\right)^{1/2}|W(R)|=1.$$

于是第二步证毕. 即定理 2,i)证毕.

ii) 由定理 B,要证(14),只证明在$\lim\limits_{T\to\infty}\dfrac{\log\frac{T}{a_T}}{\log\log T}=\infty$时

$$\overline{\lim_{T\to\infty}}\inf_{0\leqslant t\leqslant T_n-a_T}\sup_{R\in L_t(a_T)}\left(\frac{8\log\frac{T}{a_T}}{\pi^2 a_T}\right)^{1/2}|W(R)|\leqslant 1,\tag{20}$$

完全类似于 i)第一步证明可推得(20).

系 在定理 2 的条件下,有

i) 若$\lim\limits_{T\to\infty}\dfrac{\log\frac{T}{a_T}}{\log\log T}=r,0<r<\infty$,则

$$\overline{\lim_{T\to\infty}}\inf_{0\leqslant t\leqslant T-a_T}\sup_{R\in L_t(a_T)}\gamma_T|W(R)|=\sqrt{\frac{r+1}{r}}\quad\text{a.s.}\tag{21}$$

ii) 若$\lim\limits_{T\to\infty}\dfrac{\log\frac{T}{a_T}}{\log\log T}=\infty$,则

$$\lim_{T\to\infty}\inf_{0\leqslant t\leqslant T-a_T}\sup_{R\in L_t(a_T)}\gamma_T|W(R)|=1\quad\text{a.s.}\tag{22}$$

定理 3 在定理 C 的条件下,若

i) $\lim\limits_{T\to\infty}\dfrac{\log\frac{T}{a_T}}{\log\log T}=r,0<r<\infty$,则

$$\overline{\lim_{T\to\infty}}\inf_{a_T\leqslant t\leqslant T}\inf_{t\leqslant s\leqslant T}\sup_{R\in M_s(t)}\rho(T,t)|W(R)|=\sqrt{\frac{r+1}{r}}\quad\text{a.s.}\tag{23}$$

ii) $\lim\limits_{T\to\infty}\dfrac{\log\frac{T}{a_T}}{\log\log T}=\infty$,则

$$\lim_{T\to\infty}\inf_{a_T\leqslant t\leqslant T}\inf_{t\leqslant s\leqslant T}\sup_{R\in M_s(t)}\rho(T,t)|W(R)|=1\quad\text{a.s.}\tag{24}$$

证明 i) 第一步证明对$\forall\, d<\sqrt{\dfrac{r+1}{r}}$

$$\overline{\lim_{T\to\infty}}\inf_{a_T\leqslant t\leqslant T}\inf_{t\leqslant s\leqslant T}\sup_{R\in M_s(t)}\rho(T,t)|W(R)|\geqslant d\quad\text{a.s.}\tag{25}$$

为此取$\theta>1,\varepsilon>0$,使得$\dfrac{1}{d^2\theta}>\dfrac{r}{r+1}+\varepsilon,d^2\theta>1$,记$T_n=e^{e^n},K_n=[\log_\theta\dfrac{T_n}{a_{T_n}}]+1,t_k=\theta^k a_{T_n},(0<k<K_n),\varphi(n,k)=\log\dfrac{T_n}{t_{k+1}}+\log\log t_{k+1},J_{nk}=[\dfrac{T_n}{t_k}\varphi^6]+1,s_j=j\dfrac{t_k}{\varphi^6}(\varphi^6\leqslant j\leqslant J_{nk})$. 于是有

$$\overline{\lim_{T\to\infty}}B_T\overset{\triangle}{=}\overline{\lim_{T\to\infty}}\inf_{a_T\leqslant t\leqslant T}\inf_{t\leqslant s\leqslant T}\sup_{R\in M_s(t)}\rho(T,t)|W(R)|$$

$$\geqslant\overline{\lim_{T\to\infty}}\min_{0\leqslant k\leqslant K_n}\inf_{t_k\leqslant s\leqslant T_n}\sup_{R\in \overline{M}_s(t_k)}\rho(T_n,t_{k+1})|W(R)|\overset{\triangle}{=}\overline{\lim_{n\to\infty}}A_n\geqslant\underline{\lim_{n\to\infty}}A_n,$$

其中 $\overline{M}_s(t_k)=\{R:R\in M_s(t),(x_2-x_1)\vee(y_2-y_1)\leqslant\sqrt{t_k}\}$,且有

$$A_n\geqslant\min_{0\leqslant k\leqslant K_n\varphi^6\leqslant j\leqslant J_{nk}}\min\sup_{R\in\overline{M}_{s_j}(t_k)}\rho(T_n,t_{k+1})|W(R)|-6\max_{0\leqslant k\leqslant K_n\varphi^6\leqslant j\leqslant J_{nk}}\max\sup_{\substack{s_j\leqslant s\leqslant s_{j+1}\\ R\in\overline{M}_s(\sqrt{t_k(s_{j+1}-s_j)})}}\rho(T_n,t_{k+1})|W(R)|$$

$$\triangleq\min_{0\leqslant k\leqslant K_n}A_{nk}-6\max_{0\leqslant k\leqslant K_n}B_{nk}.$$

利用[2]中引理有

$$\sum_{n=1}^{\infty}P\{\min_{0\leqslant k\leqslant K_n}A_{nk}<d\}\leqslant c\sum_{n=1}^{\infty}K_ne^{-(r+\frac{1}{d^2\theta})en}n^6\leqslant c\sum_{n=1}^{\infty}e^{-(r+\frac{1}{d^2\theta})en}n^7<\infty,$$

故

$$\varliminf_{n\to\infty}\min_{0\leqslant k\leqslant K_n}A_{nk}\geqslant d\quad\text{a.s.}\tag{26}$$

另一方面,利用[4]定理 1.12.6

$$\sum_{n=1}^{\infty}P\{\max_{0\leqslant k\leqslant K_n}B_{nk}\geqslant\varepsilon\}\leqslant c\sum_{n=1}^{\infty}\frac{1}{(e^n)^4}\cdot n<\infty,$$

故

$$\varlimsup_{n\to\infty}\max_{0\leqslant k\leqslant K_n}B_{nk}=0\quad\text{a.s.}\tag{27}$$

因而

$$\varlimsup_{T\to\infty}B_t\geqslant\varliminf_{n\to\infty}A_n\geqslant\varliminf_{n\to\infty}\min_{0\leqslant k\leqslant K_n}A_{nk}-6\varlimsup_{n\to\infty}\max_{0\leqslant k\leqslant K_n}B_{nk}\geqslant d$$

因此 i)第一步证毕.

第二步证 $\varlimsup_{T\to\infty}B_T\leqslant\sqrt{\frac{r+1}{r}}$.

设 $0<r<\infty$,对$\forall\ d>\sqrt{\frac{r+1}{r}}$,取 $\varepsilon>0$,使$\frac{1}{d^2}<\frac{r}{r+1}-\varepsilon$,令 $T_n=e^n$,记 $R=[x-u,x]\times[y-v,y]$. 用$\frac{a_{T_n}}{\sqrt{T_n}}(\varphi(T_n))^{-3}$来分割区间$[0,T_n]$,各分点为

$$0,\frac{a_{T_n}}{\sqrt{T_n}}(\varphi(T_n))^{-3},2\frac{a_{T_n}}{\sqrt{T_n}}(\varphi(T_n))^{-3},\cdots,J_n\frac{a_{T_n}}{\sqrt{T_n}}(\varphi(T_n))^{-3},$$

这里 $J_n=[\frac{T_n}{a_{T_n}}(\varphi(T_n))^3]+1$. 然后考察概率:

$$P(A_n)\triangleq P\{\max_{1\leqslant j\leqslant J_{n+1}a_{T_{n+1}}\leqslant x\leqslant T_n}\inf\sup_{0\leqslant s\leqslant j\frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3}}\sup_{0\leqslant v\leqslant\frac{\sqrt{T_{n+1}}}{j(\varphi(T_n))^{-3}}}\sqrt{1+\frac{1}{r_n}\left(\frac{8\log\frac{T_{n+1}}{a_{T_{n+1}}}}{\pi^2a_{T_{n+1}}}\right)^{1/2}}|W(R)|\geqslant d\}$$

$$\leqslant\sum_{j=1}^{J_{n+}}P\{\inf_{\frac{ja_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3}\leqslant x\leqslant\sqrt{T_n}}\inf_{\frac{\sqrt{T_{n+1}}}{j(\varphi(T_n))^{-3}}\leqslant y\leqslant\sqrt{T_n}}\sup_{0\leqslant s\leqslant j\frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3}}$$

$$\cdot \sup_{0\leqslant v\leqslant \frac{\sqrt{T_{n+1}}}{j(\varphi(T_n))^{-3}}} \sqrt{1+\frac{1}{r_n}\Big(\frac{8\log \frac{T_{n+1}}{a_{T_{n+1}}}}{\pi^2 a_{T_{n+1}}}\Big)^{1/2}}|W(R)|\geqslant d\} \triangleq \sum_{j=1}^{J_{n+1}} P(A_n^{(j)}),$$

这里 $r_n=\frac{\log \frac{T_n}{a_{T_n}}}{\log\log T_n}$. 对于 $P(A_n^{(j)})$，令

$$x_i=i\Big(j\frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3}\Big),i=1,2,\cdots,p_n^{(j)},\quad p_n^{(j)}=\Big[\frac{\sqrt{T_nT_{n+1}}}{ja_{T_{n+1}}}(\varphi(T_n))^3\Big]-1,$$

$$y_k=k\Big(\frac{\sqrt{T_{n+1}}}{j(\varphi(T_n))^{-3}}\Big),k=1,2,\cdots,Q_n^{(j)},\quad Q_n^{(j)}=\Big[j\sqrt{\frac{T_n}{T_{n+1}}}(\varphi(T_n))^{-3}\Big]-1,$$

于是

$$P(A_n^{(j)})\leqslant P\{\min_{1\leqslant i\leqslant p_n^{(j)}}\min_{1\leqslant k\leqslant Q_n^{(j)}}\sup_{1\leqslant u\leqslant j\frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3}}\sup_{0\leqslant v\leqslant \frac{\sqrt{T_{n+1}}}{j(\varphi(T_n))^{-3}}}\sqrt{1+\frac{1}{r_n}\Big(\frac{8\log \frac{T_{n+1}}{a_{T_{n+1}}}}{\pi^2 a_{T_{n+1}}}\Big)^{1/2}}|W(R)|\geqslant d\}$$

$$\leqslant \exp\{-cn^\delta\},\quad \delta>0.$$

进而

$$\sum_{n=1}^{\infty}P(A_n)\leqslant \sum_{n=1}^{\infty}n^{r_n}\log n^{r_n}\exp\{-cn^\delta\}<\infty,$$

即

$$\varlimsup_{n\to\infty}\max_{1\leqslant j\leqslant J_{n+1}}\inf_{a_{T_{n+1}}\leqslant s\leqslant T_n}\sup_{0\leqslant u\leqslant j\frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3}}\sup_{0\leqslant v\leqslant \frac{\sqrt{T_{n+1}}}{j(\varphi(T_n))^{-3}}}\sqrt{1+\frac{1}{r_n}\Big(\frac{8\log \frac{T_{n+1}}{a_{T_{n+1}}}}{\pi^2 a_{T_{n+1}}}\Big)^{1/2}}|W(R)|\leqslant d. \tag{28}$$

对任一 $R=[x-u,x]\times[y-v,y]\in M_s(a_{T_{n+1}})$，有 $uv\leqslant a_{T_{n+1}},xy=s,0\leqslant x,y\leqslant\sqrt{s},0\leqslant x-u,y-v\leqslant\sqrt{s}$. 故存在一个 j 使得

$$j\frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3}\leqslant u\leqslant (j+1)\frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3},$$

即 R 可分为两部分，一部分 R_1 在矩形

$$\Big[x-j\frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3},x\Big]\times\Big[y-\frac{\sqrt{T_{n+1}}}{j(\varphi(T_n))^{-3}},y\Big]$$

中，另一部分 R_2 在矩形

$$\Big[x-(j+1)\frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3},x-j\frac{a_{T_{n+1}}}{\sqrt{T_{n+1}}}(\varphi(T_n))^{-3}\Big]\times\Big[y-\frac{\sqrt{T_{n+1}}}{j(\varphi(T_n))^{-3}},y\Big]$$

中. 于是有

$$\overline{\lim_{T\to\infty}} B_T \leqslant \overline{\lim_{s\to\infty}} \inf_{a_{T_{s+1}}\leqslant s\leqslant T_s} \sup_{R\in M_s(a_{T_{s+1}})} \left(\left(1+\frac{1}{r_{s+1}}\right)\frac{8\log\frac{T_{s+1}}{a_{T_{s+1}}}}{\pi^2 a_{T_{s+1}}}\right)^{1/2} |W(R)|$$

$$\leqslant \overline{\lim_{s\to\infty}} \max_{1\leqslant j\leqslant J_{s+1}} \inf_{a_{T_{s+1}}\leqslant s\leqslant T_s} \sup_{0\leqslant u_1\leqslant j\frac{a_{T_{s+1}}}{\sqrt{T_{s+1}}}(\varphi(T_s))^{-3}} \sup_{0\leqslant v_1\leqslant \frac{\sqrt{T_{s+1}}}{j(\varphi(T_s))^{-3}}} \left(\left(1+\frac{1}{r_{s+1}}\right)\frac{8\log\frac{T_{s+1}}{a_{T_{s+1}}}}{\pi^2 a_{T_{s+1}}}\right)^{1/2} |W(R_1)|$$

$$+\overline{\lim_{s\to\infty}} \sup_{0\leqslant x\leqslant\sqrt{T_{s+1}}} \sup_{0\leqslant y\leqslant\sqrt{T_{s+1}}} \sup_{\substack{0\leqslant u_2\leqslant\sqrt{T_{s+1}}\\ 0\leqslant x+u_2\leqslant\sqrt{T_{s+1}}}} \sup_{\substack{0\leqslant v_2\leqslant\sqrt{T_{s+1}}\\ 0\leqslant y+v_2\leqslant\sqrt{T_{s+1}}}} \left(\left(1+\frac{1}{r_{s+1}}\right)\frac{8\log\frac{T_{s+1}}{a_{T_{s+1}}}}{\pi^2 a_{T_{s+1}}}\right)^{1/2} |W(R_2')|$$

$$\stackrel{\triangle}{=} I_1+I_2, \tag{29}$$

这里 $R_2'=[x,x+u_2]\times[y,y+v_2]$，由(28)知 $I_1\leqslant\sqrt{\frac{r+1}{r}}$，由定理 2，(18)知 $I_2=0$，于是第二步证毕. 即 i)证毕. ii)要证(24)成立只要证明在 $\lim_{T\to\infty}\frac{\log\frac{T}{a_T}}{\log\log T}=\infty$ 时有

$$\overline{\lim_{s\to\infty}} \inf_{a_T\leqslant t\leqslant T} \inf_{t\leqslant s\leqslant T} \sup_{R\in M_s(t)} \rho(T,t)|W(R)| \leqslant 1. \tag{30}$$

完全类似于 i)第二步的证明可证(30)成立.

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Some results for two-parameter Wiener Process

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Abstract

In this paper we disuss some results of two-parameter Wiener process, which is related to [1]. First we find the normalizing factor μ_r such that $\overline{\lim_{T\to\infty}} \inf_{0\leqslant x\leqslant T-a_T} \inf_{0\leqslant y\leqslant T-b_T} \sup_{0\leqslant s\leqslant a_T} \sup_{0\leqslant t\leqslant b_T} |W([x,x+s]\times[y,y+t])| = 1$. Then we give $\overline{\lim}$ of more general increments and $\overline{\lim}$ of their leg increments.

Keywords two-parameter Wiener process, normalizing factor, increment, leg increment.