# 拟可微函数优化的Lagrange乘子与一阶最优性条件*

高 岩 邓明荣
(燕山大学，秦皇岛066004) (杭州大学，杭州310028)

**摘 要** 本文给出了拟可微优化的Fritz John必要条件与Shapiro最优性必要条件的等价性质以及两个最优性充分条件.

**关键词** 不可微优化，拟可微函数，最优性条件.
**分类号** AMS(1991) 49K52,90C30/CCL O224

## 1 Fritz John点与Shapiro稳定点的等价性

在无任何非退化假设下得到的不等式约束拟可微优化的一阶最优性必要条件主要有两种形式. 一是Shapiro的几何形式的最优性条件[1]；另一个是文[2,3]中利用Lagrange乘子得到的Fritz John形式的最优性条件. 考虑下述问题

$$(\mathrm{P})\qquad \begin{array}{ll} \min & f_0(x) \\ \mathrm{s.t.} & f_i(x) \leqslant 0, i=1,\cdots,m \end{array}$$

其中$f_i(x), i=0,\cdots,m$均为$R^n$上的拟可微函数(在Demyanov意义下)[4]. 将文[1]结果推广到多个约束情形有下述结果

**引理1** 设$\bar{x}\in R^n$为问题(P)的最优解，记$I(\bar{x})=\{i \mid f_i(\bar{x})=0, i=1,\cdots,m\}$，则

$$-\sum_{i\in I(\bar{x})\cup\{0\}}\bar{\partial} f_i(\bar{x}) \subset \mathrm{co}\{\underline{\partial} f_i(\bar{x}) - \sum_{j\in I(\bar{x})\cup\{0\}/\{i\}}\bar{\partial} f_j(\bar{x}) \mid i\in I(\bar{x})\cup\{0\}\}. \tag{1}$$

问题(P)中满足上式的可行点称为Shapiro稳定点.

**引理2**[2,3] (Fritz John条件) 设$\bar{x}\in R^n$为问题(P)的最优解，则对任意一组超微分$w_i\in \bar{\partial} f_i(\bar{x}), i=0,\cdots,m$，存在一组不全为零(依赖于$w_i$)的常数$\lambda_i(w)\geqslant 0, i=0,\cdots,m$，使得

$$0\in \sum_{i=0}^{m}\lambda_i(w)(\underline{\partial} f_i(\bar{x})+w_i), \tag{2}$$

$$\lambda_i(w) f_i(\bar{x}) = 0. \tag{3}$$

满足式(2)和(3)的可行点称为问题(P)的Fritz John点. 若对任意一组超微分$w_i, i=0,\cdots,m$，式(2)中$\lambda_0(w)\neq 0$，取$\lambda_0(w)=1$，此时Fritz John点称为Kuhn-Tucher点.

**引理3**[4] 设$A$为$R^n$中紧凸集，则支撑函数$\delta^*(x\mid A)=\max\limits_{v\in A}\langle v,x\rangle$为$R^n$上的凸函数，其次微分为$\partial\delta^*(x\mid A)=\mathrm{co}\{v\mid v\in A, \langle v,x\rangle=\delta^*(x\mid A)\}$，特别$\partial\delta^*(o\mid A)=A$.

* 1993年6月14日收到.

**引理 4** 设 $\bar{x}$ 为问题(P)的 Fritz John 点，则 $\forall\ d \in R^n$ 存在一组不全为零的常数 $\lambda_i(d) \geqslant 0$，$i=0,\cdots,m$ 使得 $0 \leqslant \sum_{i=0}^{m} \lambda_i(d) f_i'(\bar{x};d)$，$\forall\ d \in R^n$.

**证明** $\forall\ d \in R^n$，选取 $w_i \in \bar{\partial} f_i(\bar{x})$，$i=0,\cdots,m$ 使得 $\min_{w \in \bar{\partial} f_i(\bar{x})} \langle w, d \rangle = \langle w_i, d \rangle$. $\bar{x}$ 为(P)的 Fritz John 点，则存在一组不全为零的常数 $\lambda_i(w) \geqslant 0$，$i=0,\cdots,m$（记 $\lambda_i(d)=\lambda_i(w)$）使得 $0 \in \sum_{i=0}^{m} \lambda_i(d)(\underline{\partial} f_i(\bar{x}) + w_i)$，于是

$$0 \leqslant \max_{v \in \sum_{i=0}^{m} \lambda_i(d)(\underline{\partial} f_i(\bar{x})+w_i)} \langle v, d \rangle = \sum_{i=0}^{m} \lambda_i(d)[\max_{v \in \underline{\partial} f_i(\bar{x})} \langle v, d \rangle + \langle w_i, d \rangle]$$
$$= \sum_{i=0}^{m} \lambda_i(d) f_i'(\bar{x};d), \quad \forall\ d \in R^n. \tag{4}$$

**定理 1** $\bar{x} \in R^n$ 为问题(P)的 Fritz John 点的充要条件是 $\bar{x}$ 为问题(P)的 Shapiro 稳定点.

**证明** 必要性. 设 $\bar{x}$ 为问题(P)的 Fritz John 点，由引理 4，$\forall\ d \in R^n$ 存在一组不全为零的常数 $\lambda_i(d) \geqslant 0$，$i=0,1,\cdots,m$ 使得

$$\sum_{i=0}^{m} \lambda_i(d) f_i'(\bar{x};d) \geqslant 0. \text{ 又 } \lambda_i(d) f_i(\bar{x}) = 0, i = 1,\cdots,m,$$

于是

$$\sum_{i \in I(\bar{x}) \cup \{0\}} \lambda_i(d) f_i'(\bar{x};d) \geqslant 0, \max\{f_i'(\bar{x};d) \mid i \in I(\bar{x}) \cup \{0\}\} \geqslant 0.$$

令

$$F(x) = \max\{f_0(x) - f_0(\bar{x}), f_i(x) \mid i \in I(\bar{x})\},$$

$F(x)$ 为 $R^n$ 上拟可微函数，其在 $\bar{x}$ 处方向导数和拟微分为

$$F'(\bar{x};d) = \max\{f'(\bar{x};d) \mid i \in I(\bar{x}) \cup \{0\}\}, \underline{\partial} F(\bar{x})$$
$$= \mathrm{co}\{\underline{\partial} f_i(\bar{x}) - \sum_{j \in I(\bar{x}) \cup \{0\}/\{i\}} \bar{\partial} f_i(\bar{x}) \mid i \in I(\bar{x}) \cup \{0\}\}, \bar{\partial} F(\bar{x}) = \sum_{i \in I(\bar{x}) \cup \{0\}} \bar{\partial} f_i(\bar{x}).$$

由于 $\forall\ d \in R^n$，$F'(\bar{x};d) \geqslant 0$，则 $-\bar{\partial} F(\bar{x}) \subset \underline{\partial} F(\bar{x})$，即式(1)成立，$\bar{x}$ 为问题(P)的 Shapiro 稳定点.

充分性. 设 $\bar{x}$ 为(P)的 Shapiro 稳定点，则 $-\bar{\partial} F(\bar{x}) \subset \underline{\partial} F(\bar{x})$，于是 $F'(\bar{x};d) = \max\{f_i'(\bar{x};d) \mid i \in I(\bar{x}) \cup \{0\}\} \geqslant 0$，即系统

$$f_i'(\bar{x};d) = \max_{v \in \underline{\partial} f_i(\bar{x})} \langle v, d \rangle + \min_{w \in \bar{\partial} f_i(\bar{x})} \langle w, d \rangle < 0, i \in I(\bar{x}) \cup \{0\}$$

无解. 对任意 $w_i \in \bar{\partial} f_i(\bar{x})$，$i \in I(\bar{x}) \cup \{0\}$，系统

$$\max_{v \in \underline{\partial} f_i(\bar{x})} \langle v, d \rangle + \langle w_i, d \rangle < 0, \forall\ d \in R^n, i \in I(\bar{x}) \cup \{0\}$$

也无解. 于是下述优化问题

$$(\mathrm{P}_1) \qquad \begin{array}{ll} \min & z \\ \text{s.t.} & \max_{v \in \underline{\partial} f_i(\bar{x})} \langle v, d \rangle + \langle w_i, d \rangle \leqslant z, i \in I(\bar{x}) \cup \{0\} \end{array}$$

最优值为零. 问题$(P_1)$是以$(z,d)\in R^{n+1}$为变量的凸规划,$0\in R^{n+1}$为$(P_1)$的最优解. 记

$$z(z,d)=z,g_i(z,d)=\max_{v\in \underline{\partial} f_i(\bar{x})}\langle v,d\rangle+\langle w_i,d\rangle-z,i\in I(\bar{x})\cup\{0\},$$

根据凸规划的 Fritz John 条件存在不全为零的常数 $\bar{\lambda}_0(w),\lambda_i(w)\geqslant 0,i\in I(\bar{x})\cup\{0\}$使得

$$0\in\bar{\lambda}_0(w)\partial z(0)+\sum_{i\in I(\bar{x})\cup\{0\}}\lambda_i(w)\partial g_i(0). \tag{5}$$

利用引理 3 计算次微分得

$$\partial z(0)=(1,0),0\in R^n,\partial g_i(0)=\{(-1,v_i+w_i)\mid v_i\in\underline{\partial} f_i(\bar{x})\},i\in I(\bar{x})\cup\{0\}.$$

由式(5)

$$0\in\{(\bar{\lambda}_0(w)-\sum_{i\in I(\bar{x})\cup\{0\}}\lambda_i(w),\sum_{i\in I(\bar{x})\cup\{0\}}\lambda_i(w)(v_i+w_i))\mid v_i\in\underline{\partial} f_i(\bar{x})\},$$

于是

$$0=\bar{\lambda}_0(w)-\sum_{i\in I(\bar{x})\cup\{0\}}\lambda_i(w),0\in\sum_{i\in I(\bar{x})\cup\{0\}}\lambda_i(w)(\underline{\partial} f_i(\bar{x})+w_i),$$

此处 $\lambda_i(w),i\in I(\bar{x})\cup\{0\}$不全为零(否则 $\bar{\lambda}_0(w)$也为零). $\bar{x}$ 为(P)的 Fritz John 点.

## 2 Kuhn-Tucker 充分条件

**定义 1** 设 $f(x)$为 $R^n$ 上的方可微函数. 若对任意满足 $f'(x;x_2-x_1)\geqslant 0$ 的 $x_1,x_2$ 有 $f(x_2)\geqslant f(x_1)$,称 $f(x)$为伪凸函数. 若对任意 $x_1,x_2\in R^n,0\leqslant\lambda\leqslant 1$ 有 $f[\lambda x_1+(1-\lambda)x_2]\leqslant\max\{f(x_1),f(x_2)\}$,称 $f(x)$为拟凸函数.

**定理 2** 若对任意一组超微分 $w_i\in bd\ \bar{\partial} f_i(\bar{x}),i\in I(\bar{x})\cup\{0\}$,存在常数 $\lambda_i(w)\geqslant 0,i\in I(\bar{x})\cup\{0\}$,使得

$$0\in\operatorname{int}\{\sum_{i\in I(\bar{x})\cup\{0\}}\lambda_i(w)(\underline{\partial} f_i(\bar{x})+w_i)\}\text{ 又 }f_i(x),i\in I(\bar{x})\cup\{0\}$$

在 $\bar{x}$ 处一致方向可微[4],则 $\bar{x}$ 为问题(P)的严格局部极小点.

**证明** 用反证法. 假设 $\bar{x}$ 不是(P)的严格局部极小点,必存在点列 $x_k\to\bar{x}(k\to+\infty)$,使得 $f_0(x_k)\leqslant f_0(\bar{x}),x_k$ 可表示为 $x_k=\bar{x}+\theta_k d_k$,其中 $\|d_k\|=1,\theta_k\to 0^+(k\to+\infty)$. $\{d_k\}_1^\infty$ 有界,必有子列收敛到 $\bar{d}\in R^n$,不妨设 $d_k\to\bar{d}$. 由 $f_i(x),i\in I(\bar{x})\cup\{0\}$的一致方向可微性及

$$f_i(x_k)\leqslant f_i(\bar{x}),i\in I(\bar{x})\cup\{0\},$$

有[4]

$$0\geqslant f_i(x_k)-f_i(\bar{x})=f_i(\bar{x}+\theta_k d_k)-f(\bar{x})=\theta_k f_i'(\bar{x};\bar{d})+o(\theta_k),\quad i\in I(\bar{x})\cup\{0\}, \tag{6}$$

于是 $f_i'(\bar{x};\bar{d})\leqslant 0,i\in I(\bar{x})\cup\{0\}$. 选取 $w_i\in\bar{\partial} f_i(\bar{x})$,使得 $\min\limits_{w\in\bar{\partial} f_i(x)}\langle w,\bar{d}\rangle=\langle w_i,\bar{d}\rangle$,由

$$0\in\operatorname{int}\sum_{i\in I(\bar{x})\cup\{0\}}\lambda_i(w)(\underline{\partial} f_i(\bar{x})+w_i)\}$$

得

$$0<\max_{v\in\sum_{i\in I(\bar{x})\cup\{0\}}\lambda_i(w)(\underline{\partial} f_i(\bar{x})+w_i)}\langle v,\bar{d}\rangle=\sum_{i\in I(\bar{x})\cup\{0\}}\lambda_i(w)f_i'(\bar{x};\bar{d})\leqslant 0, \tag{7}$$

矛盾.

**定理 3** 设 $\bar{x}\in R^n$ 为问题(P)的 Kuhn-Tucker 点，$f_0(x)$为伪凸函数，$f_i(x)$，$i\in I(\bar{x})$为拟凸函数，则 $\bar{x}$ 为(P)的最优解.

## 参 考 文 献


[1] A. Shapiro, *On optimality conditions in quasi-differentiable optimization*, SIAM J. Control and Opt., 4(1984), 610—617,

[2] 高 岩，拟可微函数的弱 Fritz John 条件，运筹学杂志，2(1990)，43—44.

[3] B. Luderer, *Directional derivative estimates for the optimal value function of a quasi-differentiable programming*, Math. Prog., 3(1991), 333—348.

[4] V. F. Demyanov, A. M. Rubinov, *Quasidifferential Calculus*, Optimization Software Inc., New York, 1986.


# The Lagrange Multiplier and First Order Optimality Conditions for Quasidifferentiable Optimization


*Gao Yan* (Yanshan University, Qinhuangdao 066004)

*Deng Mingrong* (Hangzhou University, 310028)



**Abstract**

The equivalence of Firtz John necessary condition and shapiro's necessary condition for quasidifferentiable optimization is proved. Two sufficient optimility conditions for quasidifferentiable optimization are given.



**Keywords** nondifferentiable optimization, quasidifferentiable function, optimility condition.